Article

Mindnature: Origin of Physicality & Mathematics

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Abstract

The existence of the irrational numbers indicates that reality is not a structure of inherent existence; it is a structure within emptiness. In other words it is because of the background fluid and indeterminate nature of emptiness that any reality can function at all, a remarkable insight on the part of Buddhist philosophy dramatically verified by quantum physics. And this ultimately 'empty' nature is revealed by the very fact that such fluidly precise and yet in a sense ungraspable conceptual procedures have to be employed within mathematical analysis. If both the realm of mentality and physicality emerge from a deeper level of universal Mindnature then it is surely not such a great mystery that mathematics, itself a product of mind, produces the conceptual patterns generated and followed by the 'physical' functioning of reality.

Keywords: mathematics, emptiness, illusion, mind, Gödel, Penrose.

In his magnum opus *The Road to Reality* Roger Penrose, after the obligatory brief introductory description of how our bewildered ancestors conceived the functioning of the universe to be due to the activities of gods, tells us that they needed to:

> …discover how to disentangle the true from the suppositional in *mathematics*. A procedure was required for telling whether a given mathematical assertion is or is not to be trusted as true.[1]

Penrose's basic viewpoint is that there is an 'objective' sphere of Platonic mathematical truth, a realm of logical and mathematical precision which 'exists' independently of individual human minds. He presents this pristine realm of crystalline mathematical certitude as an ideal sphere of perfect precision and his invocation of this ethereal mansion of mathematical rectitude is striking:

> It tells us to be careful to distinguish the precise mathematical entities from the approximations that we see around us in the world of physical things. Moreover, it provides us with the blueprint according to which science has proceeded ever since. Scientists will put forward models of the world … The models are deemed to be appropriate if they survive such rigorous examination and if, in addition, they are internally consistent structures. … The required precision demands that the model be a mathematical one, for otherwise one cannot be sure that these questions have well defined answers.[2]

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Penrose further suggests that there are three separate worlds, the physical, the mental and the platonic-mathematical. The actual interrelations between these three are that the physical gives rise to the mental, the mental somehow maps on to the platonic-mathematical and a portion of the platonic-mathematical somehow maps on the 'physical'. The details of this set of relations are sketchy, but the important feature for our purposes for the moment is that the realm of mathematical precision is invested with a kind of pristine purity that accords it a special status as the primary 'road to reality.'

This, of course, is the kind of picture that is often conveyed to students and intellectual consumers in general within our scientific culture. Mathematics is the razor sharp means of dissecting and analyzing reality with a precision so accurate that every hinge and joint of reality is analyzed and understood to an ultimate level of precision. This, of course, is to a large extent remarkably true. Although actual physical reality has rough edges, so to speak, the glittering, glinting and immaculate lineaments of the independent realm of mathematical structures and truths fits over and illuminates the functioning of actual reality with a precision that we can only gasp and wonder at. The physicist Eugene Wigner, for instance, has referred to what he considers to be the 'unreasonable effectiveness' of mathematics in describing and explaining the physical world of 'nature'; he calls mathematics a 'miracle' and 'a wonderful gift that we neither understand nor deserve.'[3] In this paper, however, we will find that the effectiveness of mathematics may not be as mysterious as Wigner thinks. The reason that mathematics is so handy for analyzing and describing the functioning of the 'physical' world is that both the 'mental' and 'physical' realms have an origin in a deep level of Mind which underlies all phenomena.

In the late nineteenth and early twentieth centuries it was thought by many that the precision and effectiveness of mathematics was based upon firm logical foundations. The work of Frege, Russell, Whitehead and Hilbert was aimed at elucidating and providing the logical framework which would prove that mathematics was indeed based on solid logical foundations. Russell and Whitehead set out to provide the logical foundations for mathematics in a work entitled *Principia Mathematica.* However, in 1931 the genius logician Kurt Gödel published "Über Formal Unentscheidbare Sätze der *Principia Mathematica* und Verwandter Systeme" (translated into English "On Formally Undecidable Propositions of *Principia Mathematica* and Related Systems"). In that article, he proved for any axiomatic system that is powerful enough to describe the arithmetic of the natural numbers, the kind of axiomatic system that Russell and Whitehead wanted to develop, that such a project was not viable. Gödel's incompleteness theorems ended the illusion that all mathematical truths could be contained and proved within one consistent and complete axiomatic system. The mathematician Ian Stewart tells is that:

> After Gödel, mathematical truths turned out to be an illusion.[4]

We shall get to Gödel's theorem towards the end of this paper, but the problems for Penrose's rose tinted perspective regarding ideal realms of mathematical absolute pristine truth and precision actually began much further back in the historical development of mathematics.

The fact that Penrose is well aware of the relevant issue, although he attempts to underplay its dramatic significance, is indicated in his section heading *A Pythagorean catastrophe,* the heading under which he discusses it. The Greek Pythagorean philosophers had a very reverent attitude towards numbers. In his excellent book *Infinity: The Quest to Think the Unthinkable* Brian Clegg describes their perspective as follows:

> The Pythagoreans considered numbers to be among the building blocks of the universe. In fact, one of the most central of the beliefs of Pythagoras' *mathematikoi*, his inner circle, was that reality was mathematical in nature.[5]

The Pythagorean conception of the relationship between the realm of number and reality found its exemplary image in the Pythagorean perception of the relationship between number and geometrical figures. Stewart describes the fundamental attitude to the notion of number as being embodied in the view that:

> They considered the number 1 to be the prime source of everything in the universe.[6]

And, according to Tobias Dantzig, the Pythagoreans had a corresponding conception of the geometric point:

> The point is unity in position' was the basis of Pythagorean geometry. Behind this flowery verbiage we detect the naïve idea of the line as made up of a succession of atoms as a necklace is made up of beads.[7]

In other words the Pythagoreans believed in the inherent existence of both numbers and geometric points. The term 'inherent existence' is taken from Buddhist philosophy; it indicates a belief in structures of reality which have their own internal 'inherent' reality which is completely independent of other phenomena, including the minds of perceivers. The Pythagoreans, then, believed in an inherently existent reality which was comprised of geometrical entities which were in turn comprised of 'partless' ultimate points; and this sharply defined structure of the physical world was thought to be mirrored by the perfect geometric figures of geometry within which there could be found inherently existent 'rational' numerical relationships.

This view of the makeup of reality is crudely illustrated in fig 1. This is the famous Pythagorean 3-4-5 right-angled triangle. The sides of this triangle fit the theorem of Pythagoras that:

$$5^2 = 3^2 + 4^2$$

$$5x5 = 3x3 + 4x4$$

$$25 = 9 + 16$$

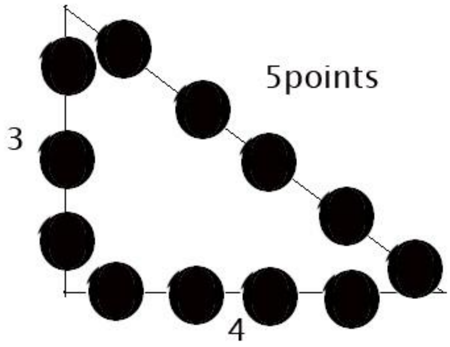

Fig 1

The crucial point is that the Pythagorean view of the situation required that the number of points which fitted along the sides of the triangles should be such that the lengths could be expressed as rational numbers, which are numbers of the form:

$$N / M$$

(N divided by M) where N and M are whole numbers. Triangles of this form can always be scaled up so that there is a whole number of points along each side just like the 3-4-5 triangle.

This Pythagorean fantasy of an inherently existent geometric atomism which was thought to be echoed in the mathematical structures which echo the geometric world held sway for a while until someone spoiled the party of Pythagorean purity by discovering something very disturbing about unit squares, which are squares which have all sides of length 1. According to Pythagoras's theorem the diagonal of the unit square must have a length which is the square root of 2 (**√2**) (fig 2). The problem for the pristine purity of the Pythagorean geometrical-numerical perspective arises because it can be proved that number **√2** cannot be represented as a rational number of the form M/N.

The proof by contradiction that is now used for showing that √2 is not a rational number is so sweet it is worth outlining for those not familiar with it. We start by assuming that √2 is a rational number. Then we can write it √2 = M/N where M, N are whole numbers, N not zero. We additionally require that M/N is reduced to lowest terms, having no common factors, which can obviously be done with any fraction. Now by squaring both sides we get $2 = M^2N^2$, or $M^2 = 2N^2$. So the square of M is an even number. From this it follows that M itself is also an even number. Why? Because it can't be odd; if M itself was odd, then (M x M) would be odd too because an odd number times an odd number is always odd. If M itself is an even number, then M is 2 times some other whole number, or M = 2k where k is some other number. If we substitute M = 2k into the original equation $2 = M^2/N^2$, this is what we get:

$$2 = (2k)^2/N^2$$
$$2 = 4k^2/N^2$$

$$2N^2 = 4k^2$$
$$N^2 = 2k^2.$$

This means N^2 is even, from which follows again that N itself is an even number! So we have now derived a contradiction. This is because we started the analysis by assuming that M/N is reduced to the lowest terms, but it now it turns out that M and N would both be even. So √2 cannot be rational.

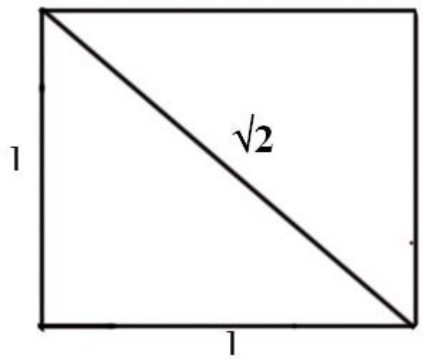

Fig 2

For the Greeks this was a completely new kind of number, as we shall see very shortly it is a kind of non-inherently existent (using the notion of non-inherence to indicate a lack of the kind of precise boundaries which are characteristic of natural numbers), or *irrational*, number which has some very disturbing characteristics from the perspective of anybody who prefers numbers to come in inherently existent, or rational, flavors. The Pythagoreans were so disturbed by the shattering of their inherently existence numerical world that Proclus wrote that:

> It is told that those who first brought out the irrationals from their concealment into the open perished in a shipwreck to a man. For the unutterable and the formless must needs be concealed.[8]

For Proclus and other Greek philosophers, then, the discovery of irrational numbers opened up a view upon disconcerting vistas of 'the unutterable and the formless'. Within Buddhist philosophy the term employed to indicate the lack of 'inherent existence' in all phenomena is 'emptiness' (*shunyata*). This term does not indicate 'nothingness' as usually understood as a complete and absolute void, it indicates, rather, no-thing-ness, an ungraspable realm of indeterminate potentiality, an indeterminate realm from which all things emerge as illusion-like temporary appearances.

The Buddhist notion of 'emptiness', then, is clearly connected to the twentieth century discoveries within quantum physics, as quantum physicist Vlatko Vedral says:

> Quantum physics is indeed very much in agreement with Buddhistic emptiness.[9]

Emptiness, or *shunyata,* is the Buddhist concept of a fundamental non-substantial 'empty' ground of potentiality which gives rise to the multitudinous productions within dualistic experience through the operation of a primordial activity of cognition. The infinite acts of primordial cognition which drive the process of dualistic reality, and form the basis for the higher level cognitive continuums of sentient beings, can be considered as internal activations of creation operators within the fundamental quantum field. And, just as the fundamental quantum field itself lacks substantiality, so too, when analyzed thoroughly, it turns out that all phenomena arising from the primordial cognitive activations within the quantum field equally lack absolute and independent substantiality. Or, as Buddhist philosophy of the Madhyamaka, the Middle Way analysis, asserts, all phenomena lack, or are 'empty' of 'inherent existence' (*svabhava*).

The seemingly 'physical' world within which we have our embodied being, however, seems remarkably substantial, it certainly appears to have 'inherent existence'. And, because the world does present such a convincing appearance of materiality and solidity, the Madhyamaka, a dazzling metaphysical deconstruction of our notions of everyday reality founded by the Indian Buddhist philosophical genius Nagarjuna in the 2nd century C.E., employs deconstructive analyses of our concepts of, and the apparent functioning of, reality in order to show the illusion-like nature of appearances.[10] The process is reminiscent of an aspect of the film *The Matrix* wherein the minds of human beings are trapped within a vast virtual reality generated by a computer whilst their bodies are used to generate energy. The central protagonist Neo is made aware of the illusion-like nature and is thereby also becomes aware of 'glitches' in the programming of the matrix. The Madhyamaka analysis displays 'glitches' within the way in which we think reality functions in order to give us insight into the impossibility of our familiar notions of the functioning of the world. We too, then, can become aware of the illusion-like nature of reality. In this paper we are concerned with the glitches in our notion of mathematics as applying to and describing an inherently existent world.

The irrationality of **√2** and irrational numbers in general may be considered one such glitch. So what is it about the number **√2** which makes it 'unutterable' to the extent that uttering it invites shipwreck? The answer is, although the Greeks did not view the issue in exactly this way, that it is a number which can only be represented by an unending numerical expression. As a decimal expression we would have to write:

$$\sqrt{2} = 1.4142135623730950488021688 72\ldots$$

The … means that this expression never ends, it goes on for ever and ever, all the way to infinity, and, as Clegg tells us, infinity is:

> a fascinating, elusive topic. It can be like a deer, spotted in the depths of a thick wood. You will catch a glimpse of the beauty that stops you in your tracks, but moments later you are not sure if you saw anything at all. … We may then open up clear views on this most remarkable of mathematical creatures – a concept that goes far beyond sheer numbers, forcing us to question our understanding of reality.[11]

The fact that the length of the diagonal of a unit square is an irrational number means that a length of line which anyone can construct quite quickly with a ruler, pencil and right-angled triangle does not have a precise finite decimal representation. The diagonal line inside the square quite obviously has a definite end point but the decimal representation does not! As Clegg points out, this situation seriously undermines the Pythagorean conception that the innermost functioning of reality is entirely dependent on the crystalline precision of numbers:

> That handy Pythagoras' theorem about the length of sides of a right-angled triangle produces a result that is frankly devastating if you believe that the universe is driven by pure whole numbers.[12]

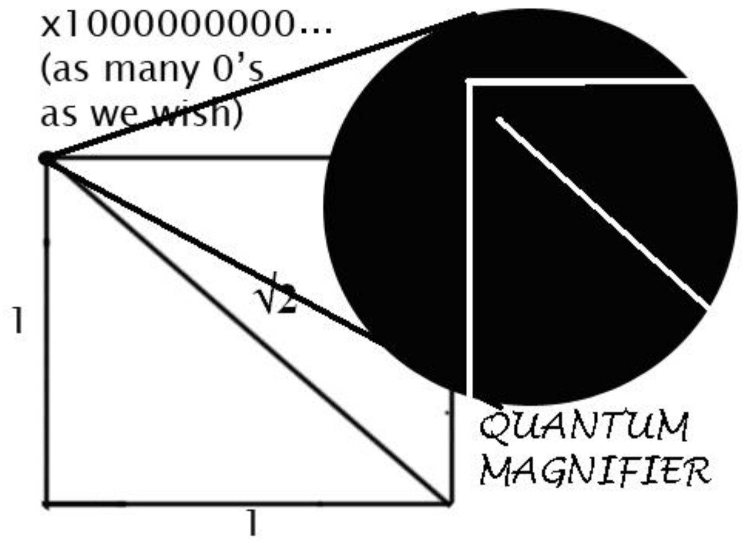

Fig 3

Just to make sure the situation is fully appreciated please refer to fig 3. We imagine that, perhaps in a special Platonic realm, we have a special quantum precision pencil which allows us to draw any fractional decimal amount at any scale approaching the quantum realm, and even descending beyond that. We also have a quantum magnifier so that we can examine the vertex of the square at which we are trying to draw in, and thereby complete, the diagonal. At every point of the decimal expansion for the number **√2** we use the magnifier and pencil to draw in the next bit of the diagonal. We have been at it for eons but, although the end is in sight, it is also infinitely beyond reach. When the gap seems about to be closed, the quantumly magnified perspective shows that we are still short of the completion; whenever we draw in the next decimal bit the magnifier shows us there is still another decimal bit to go. The task seems, indeed it is, endless; in fact the task is infinitely endless!

No wonder the Pythagoreans were freaked out; this is surely an absurd situation which screams implausibility. This fact of the extraordinary mismatch between the way the world seems or appears to be constructed in the everyday 'conventional' world, and the way that it actually does exist under mathematical analysis, offers a beautiful illustration of the

Madhyamaka notion of the two truths - the relative, seeming, or conventional truth and the ultimate truth. As we shall see it is also an excellent analogy for, or sign of, emptiness. The definition of conventional reality is 'an appearance within experience which satisfies as being real as long as it is not analyzed.' Such truths, however, are said by the Madhyamaka to be deceiving because they cannot exist in the way that they appear to. This deceptive nature is revealed when conventional truths are subjected to 'ultimate' analysis.

In the case of the number $\sqrt{2}$, conventional reality is represented by the drawing of the unit square with the diagonal drawn in. According to Pythagoras' theorem the diagonal is definitely of length $\sqrt{2}$. Not only this, but also the diagonal line can, on the conventional level, definitely be seen to have a definite length. However, a mathematical analysis, which in this case we can take to be representative of an ultimate analysis, shows that this definite and conventionally determinate length is impossible because, the diagonal line can never reach its correct extent if the attempt is made in non-infinite time periods; no matter what length is added to the end of the decimal expansion it will either overshoot, or undershoot, the end of the line. In other words, the 'ultimate' mathematical analysis shows that the conventional drawing of the diagonal line must be an illusion; the line does not inherently exist even though we can draw it!

This example is just one amongst an infinite number of possibilities! Another is the fundamental issue of the relationship between the diameter and the circumference of a circle; the diameter (d) is the distance from one side of a circle to the other across the centre, and the circumference (c) is the distance measured around the circle itself (fig 4). The relationship between these two distances is expressed by the equation:

$$c = \pi d$$

This means that the circumference can be calculated by multiplying the diameter by the value π, which is a transcendental and irrational number.

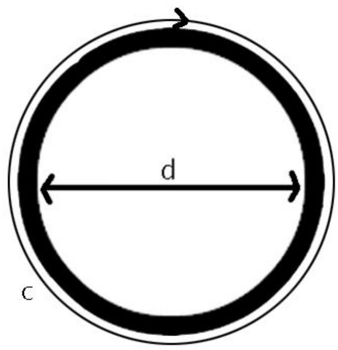

Fig 4

The fact that π is an irrational number means that, like $\sqrt{2}$, it cannot be expressed as a rational number of the form M/N, although π is *approximated* by the rational number 22/7. And when π is expressed as a decimal it begins:

3.14158…

and continues, we are told, until infinity. The irrational number π, however, is different to √2 in a very significant respect; it cannot be expressed as a term within a finite algebraic equation of any kind, including squares or other powers of an unknown entity. If we let **X,** as is mathematically usual, represent an 'unknown' value, then in the equation:

$$\mathbf{X^2 = 2} \qquad (1)$$

which means that **X** multiplied by itself gives the result of 2, the 'solution' is represented by:

$$\mathbf{X = \sqrt{2}.} \qquad (2)$$

This means that **√2** is the solution to, and is therefore derived from, equation (1). The fact that π can not be derived from such an equation is indicated by saying that π is a *transcendental* number. Clegg says of such numbers:

> Whereas **√2,** and every other irrational number that can be defined with an equation, is called algebraic to echo this property, π is far and above the best known transcendental number, the name given to irrationals that can't be fitted into a suitable finite equation. Just as irrational does not suggest lacking rational thought, transcendental has nothing to do with the mystical associations the word has picked up in the last few years. It merely says that the number transcends – is outside of – calculation by equation.[13]

Although Clegg is technically correct to attempt to undermine the notion that 'transcendental' numbers might have any 'mystical' connotations, from the point of view of our search for an inherently existent reality such a connection at least has a suggestive force. The fact that the details of the domain of the transcendental and irrational numbers, in relationship with the more well behaved and familiar realm of 'natural' and 'rational' numbers, has an affinity with the division between quantum reality and the macroscopic level of everyday life is suggested by Clegg himself:

> …modern considerations of infinity shake up the comfortable, traditional world in the same way that physicists suffered after quantum mechanics shattered the neat classical view of the way the world operated.[14]

And this attribution is correct. The determinate experienced world of dualistic appearance emerges from the quantum realm in a very similar manner to way in which the, apparently, absolute crystalline precision of the mathematical structures of meaning derive from the emergence of form from the infinite formless 'swarming', as the philosopher Alain Badiou calls it[15], of natural, rational, irrational, real, transcendental, transfinite, incomputable, and surreal numbers.

Whether such a remarkable interpenetration and interdependence between two realms of phenomena which have such antithetical and incompatible natures, is 'mystical' or not, however, clearly depends upon the meaning of the term 'mystical'. Clegg, like so many Western thinkers who do not give serious consideration to the *ultimate* implications of the issues they are dealing with, implicitly, and illicitly, implies that the 'mystical' is allied to the notion of 'lacking rational thought'. A dictionary definition, however, is:

> Having an import not apparent to the senses nor obvious to the intelligence; beyond ordinary understanding.[16]

If this definition does not apply to the fact that the ultimate mode of existence of the entities that are supposed to underlie the precise geometric figures of the conventional world, actually indicate that those precise geometric figures should not be possible, then it is difficult to conceive of just what the definition could apply to!

The forms of 'ordinary understanding' which 'are apparent to the senses' and are 'obvious to intelligence', if by this we mean ordinary embodied non-analytical intelligence, are the forms of 'inherent existence' which are those of what the Buddhist Madhyamaka calls the conventional, seeming reality. And it appears that it is exactly these comfortable appearances of the ordinary world which are shown by an ultimate mathematical analysis to be at least questionable, if not impossible.

This situation is dramatically demonstrated by the impossibility of squaring the circle. This mathematical conundrum poses the question of whether we can take a circle of a given area (A) and transform it into a square of precisely the same area; we can imagine taking the appropriate four points and then pulling outwards in order to effect the transformation (fig 5). Surely everything that is conventionally 'apparent to the senses' and 'obvious to the intelligence' would tell us that this transformation should be no problem. An 'ultimate' mathematical analysis, however, indicates that this is not the case. The number π has been shown to be definitely transcendental, so it cannot be caught within an algebraic expression. Suppose the diameter of the circle is **d**, then its area is:

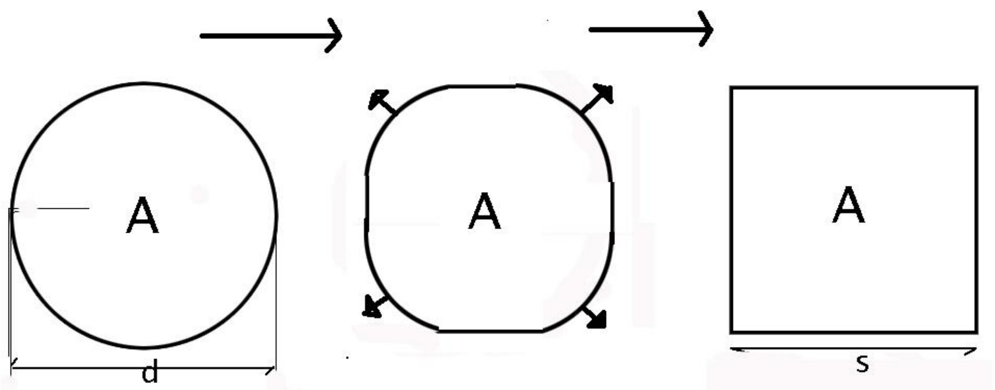

Fig 5

$$A = \pi d$$

And if the side of the square is **s**, then its area is:

$$A = s^2$$

And therefore, because the areas are supposed to be the same, then if the areas of the circle and square were to be the same then it would follow that:

$$\pi d = s^2$$

This means that if the areas of the square and circle were to be equal then it would clearly follow that π can be expressed in an algebraic expression; but the German mathematician C. L. Lindemann, in 1882, proved that π could not be so expressed. It follows, therefore, that it is impossible to perform the transformation, precisely and ultimately, of the circle into a square of equivalent area!

If we view mathematics from the point of view of inherent existence there are many such mismatches, dissonances, misalignments and so on, between the precise ultimate mathematical analysis and the functioning of the conventional realm. You may wonder how this remarkable fact is covered up, so to speak! It's easy; you just invoke a mystical realm of pure platonic logical reality! Here's what Penrose says about the strange fact that numbers like $\sqrt{2}$, irrational numbers, seem to be constructible with pencil, ruler and right-angle but cannot be finitely represented in decimal form:

> Nowadays, we do not worry unduly if a certain geometrical quantity cannot be measured simply in terms of rational numbers alone. This is because the notion of a 'real number' is very familiar to us. Although our pocket calculators express numbers in terms of only a finite number of digits, we readily accept that this is an approximation forced upon us by the fact that the calculator is a finite object. We are prepared to allow that the ideal (Platonic) mathematical number could certainly require that the decimal expansion continues indefinitely.[17]

Now this appeal to 'familiarity' is an extraordinarily lax, philosophically speaking, observation. Here we are, at the outset of Penrose's *Road to Reality*, by which he must surely mean *ultimate* reality, and at this very point we meet an extraordinary situation with regard to the nature of mathematical reality. A pencil line which, seemingly, can easily be drawn should not, from an ultimate analytical point of view, be able to be so drawn. Might this not offer us a clue as to the relationship between appearance and ultimate reality?

Penrose tells us that the notion of a 'real number is very familiar to us'. So what does this remarkable familiarity amount to? The answer is simply that mathematicians are 'familiar' with the employment of a logical fudge in order to make the situation look viable. The notion that they employ is that of a limit. This idea is enshrined in the famous paradox of Zeno regarding the possibility of crossing a definite interval when we consider the task from the point of view of traversing half of the remaining distance at each step (fig 6).

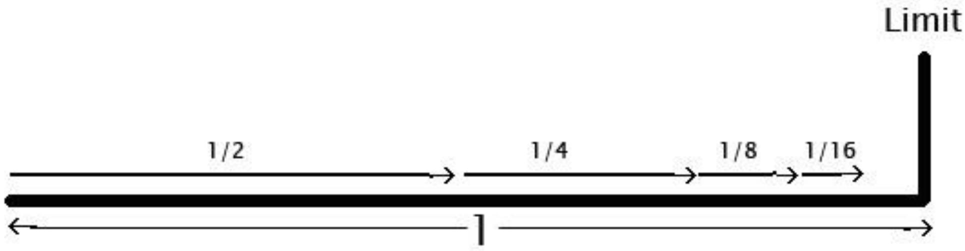

Fig 6

The problem now becomes one of finding the solution of the infinite series:

Limit (which is 1) = ½ + ¼ + 1/8 + 1/16 + 1/32 + 1/64 + 1/124 + …

To actually perform such a never ending task, of course, would take, well, forever; so what is needed is a piece of mathematical magic. And this is exactly what is done; a magical symbol which means, in essence, ‘do the impossible,’ if that is we live in an inherently existent universe, was invented (fig 7).

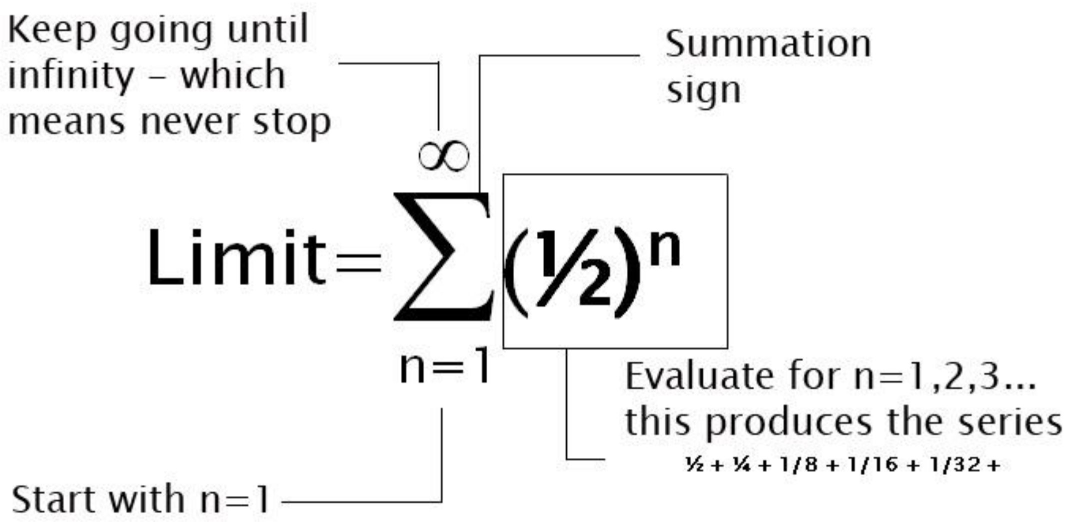

Fig 7

Now it is quite easy to show that for n = 1, 2 , 3 we have

n = 1 Limit = ½ = ½

n = 2 Limit = ½ + ¼ = ¾

n = 3 Limit = ½ + ¼ + 1/8 = 7/8

and so on

In fact the limit sum can, admittedly not so easily, be shown to be calculated by the formula:

$$\text{Limit} = 1- (½)^{n}$$

Where $(½)^1 = ½$; $(½)^2 = ¼$; $(½)^3 = 1/8$, etc.

Now we can see that the bigger the value of **n** gets, the smaller the value of $(½)^{\mathbf{n}}$ becomes. In fact the value of $(½)^{\mathbf{n}}$ gets very, very small extremely quickly. When **n**=10, for instance, $(½)^{\mathbf{n}}$ is approximately 0.001 and as **n** gets larger the value of $(½)^{\mathbf{n}}$ becomes vanishingly small. So *vanishingly small,* in fact, mathematicians have simply decided to let it *vanish*. This decision is expressed by using another mathematical magic symbol:

$$\prod_{n=1}^{\infty}(½)^{n} = 0$$

This says, mathematically speaking, that if you multiply ½ by itself an infinite number of times (∞), something which is actually impossible, but we can 'imagine' doing it (can we?), so to speak, then the result is nothing. This is indeed magic; we can start out with something and mathematically create nothing. This is something only the infinite can do! And because of this vanishing trick we can arrive at the desired result:

$$\text{Limit}(n\rightarrow\infty) = 1- (½)^{\infty} = 1- 0 = 1$$

This result follows as long as we accept that the imaginary process of multiplying a number which is less than 1 by itself an infinite number of times (∞) , a process which, in reality so to speak, can never be carried out, logically gives a zero result. And this kind of reasoning, the employment of procedures which actually are imaginatively precise manipulations of the imprecision of infinity; infinite imprecision tamed, as it were, to ever more infinitely precise tiny realms of imprecision, which underlies a great deal of the advanced techniques of modern mathematics. This is precisely why Ian Stewart entitled his recent book about the foundations of mathematics *The Taming of the Infinite*.

But it all works; *the extraordinary manipulation of the physical world which is embodied in modern physical science is based upon an incredibly precise, magical, mathematical sleight of mind!* As Clegg says 'there seemed to be a fudge in calculus'[18]; and calculus is the fundamental mathematics of change which underlies a vast amount of modern physical theory. And this 'fudge' involves a dramatic attribution of ultimate inherent existence to ultimately non-inherently existent entities: the infinitely large and the infinitesimally small. Bishop Berkeley wrote of the use of the infinitesimally small disappearing values, which were called 'fluxions' in his time:

> And what are these fluxions? The velocities of evanescent increments. And what are these same evanescent increments? They are neither finite quantities nor quantities infinitely small, nor yet nothing. May we not call them the ghosts of departed quantities?[19]

Newton wrote that:

> It may be objected that there is no such thing as an ultimate proportion of vanishing quantities, inasmuch as before vanishing the proportion is not ultimate, and after vanishing it does not exist at all … But the answer is easy … the ultimate ratio of vanishing quantities is to understood not as the ratio of quantities before they vanish or after they have vanished, but the ration with which they vanish.[20]

But in fact, when you think about it precisely, they are 'empty' entities. This is because, as Bishop Berkley indicates, they neither exist nor do they not-exist. As the Buddhist Madhyamika philosopher Bhavaviveka (1st-2nd century) indicated the 'empty' character of reality is that it is:

> Neither existent, nor nonexistent
> Nor both existent and nonexistent, nor neither.[21]

A great deal of modern mathematics, then, is redolent of emptiness!

This also means that modern mathematics is based upon irrationality, not of thought of course, but of number. Penrose's strategy of claiming some kind of pristine mathematical realm, within which irrationality can be tidied up, is suspect because the existence of irrational numbers clearly indicates something deeply unexpected about mathematical reality which, as we shall see, also means that they indicate something highly significant about reality in general. The Pythagorean Greeks did not organize shipwrecks for any minor inconvenience; they were reserved for the occurrence of major disruptions in their conception of reality. And the fact that they considered the discovery of the irrational numbers to be such a major disruption might not indicate ancient ignorant naivety, on the contrary it might be that they actually thought deeply about the implications for the nature of the reality they inhabited, rather than tidying them away from the nature of the reality that they *thought* they inhabited. The existence of the irrational numbers indicates that reality is not a structure of inherent existence; it is a structure within emptiness. As the Heart Sutra says 'form is emptiness and emptiness is form.'[22]

In other words it is because of the background fluid and indeterminate nature of emptiness that any reality can function at all, a remarkable insight on the part of Buddhist philosophy dramatically verified by quantum physics. And this ultimately 'empty' nature is revealed by the very fact that such fluidly precise and yet in a sense ungraspable conceptual procedures have to be employed within mathematical analysis. In his discussion of the concept of velocity the physicist Giovanni Vignale, in his recent book *The Beautiful Invisible,* comments that:

> Scientific theories, like works of art, live in tangential realities that are conjured up by a limiting process. Starting from familiar concepts we dive into a fantastic space,

> navigate it according to certain rules, and re-emerge on the level of reality with a new concept, a new figure of thought – velocity in this case.[23]

However, the apparent inherent reality of the physical world misleads many scientists and philosophers to impute inherent existence to conceptual tools which, as Vignale points out, are conjured out of the 'fantastic space' of conceptual imagination, and it is within these necessary conceptual 'fictions' wherein physics and mathematics offer insights into the metaphysical depths of reality. Quantum physics has already given us an 'experimental metaphysics'[24] which shows us the existentially indeterminately fluid and 'empty' ground of reality, and within mathematics we can also find quantum points of dislocation, such as the irrational, undrawable line, which offer us deep insight when we know how to look.

However, because of an ingrained belief in the inherent 'physical' existence of our reality, an inherent existence which was always conceived of as independent of Mind, many philosophers are willing to ignore points of dislocation. Penrose, for example, clearly sees the points of dislocation but seems to be beguiled by the illusion that the universe must be inherently real in nature. Perplexed observations such as 'but can a *real* world *really* be constructed on the basis of *unreal* constituents' can be found in many places in his works[25]. In fact the notion of inherent existence, the conviction that there can be completely self-enclosed, independent, self-sufficient aspects of 'reality', seems to occupy a fundamental role in Penrose's thought:

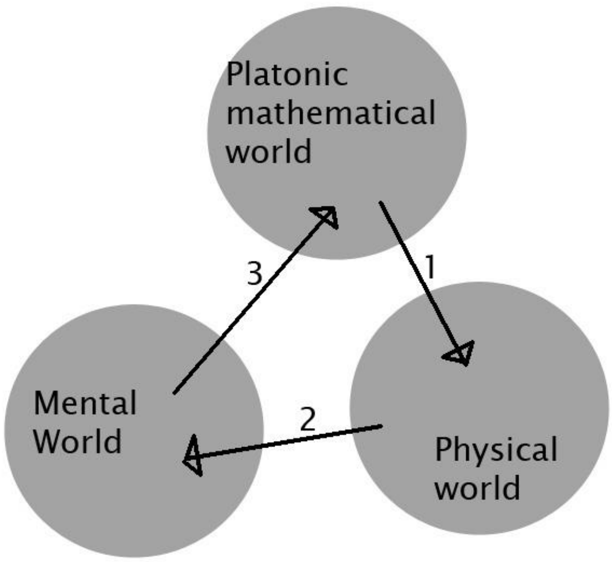

Fig 8

> …an important element in the mathematician's common conviction that an external Platonic world actually has an existence independent of ourselves comes from the extraordinary unexpected hidden beauty that the ideas themselves so frequently reveal.[26]

This notion is enshrined in his doctrine of the three worlds (fig 8). However, once having posited these realms as being fundamentally, substantially so to speak, disconnected from one another, Penrose, unsurprisingly, runs into the three 'profound mysteries' of the nature of the interconnections between the worlds. Undaunted by this crucial issue, however, Penrose presses on towards his ultimate vision of the fundamental mathematical nature of reality:

> How do I really feel about the possibility that all of my actions, and those of my friends, are ultimately governed by mathematical principles of this kind? I can live with that. I would, indeed, prefer, to have these actions controlled by something residing in some such aspect of Plato's fabulous mathematical world…[27]

But the conceptual dissection of reality into spheres, realms, or 'worlds' that are conceived of as being absolutely distinct in nature, self-enclosed, 'Platonic', sufficient unto themselves, and so on, leads automatically to incoherence exactly because aspects of reality which have nothing common which is inherent to each of them cannot possibly co-here together!

What could possibly provide a ground of commonality between such seemingly disparate aspects of reality? The simple, obvious answer is that we need to see that these aspects, which have been imputed by mind as being inherently separate from each other, have a common nature already inherent within them, they are, in the last analysis, all aspects of Mind. They are actually interdependent aspects within the unitary process of Mind, and such aspects are, in actuality, neither absolutely the same, nor absolutely different:

> When something originates in dependence upon something else,
> The depender is not the same as the depended-on,
> Nor is it different from it.[28]

This of course is the realm of emptiness; it is also the realm of the transcendental numbers! When the inner contradictions generated by the kind of, inherent-existence based, approach adopted by Penrose are penetrated and resolved, the metaphysical structure of reality reveals itself as an interdependent play of Mind appearing to itself in various guises: transcendent, rational, irrational, physical… And this metaphysical perspective emerges from a rigorous investigation of the 'empty' nature of mathematics.

The following is a distillation of Penrose's ultimate musings upon ultimate reality after he has traversed his thousand page road to reality:

> My own position on the matter is that we should certainly take Plato's world as providing a kind of 'reality' to mathematical notions …, but I might baulk at actually attempting to actually *identify* physical reality with the abstract reality of Plato's world. I think that Fig. 34.1 [see my fig 9] best expresses my opinion on this question, where each of the three worlds … has its own kind of reality, and where each is (deeply and mysteriously) in the one that precedes it (the worlds being taken cyclically). I like to think that, in a sense, the Platonic world may be the most primitive of the three, since mathematics is a kind of necessity, virtually conjuring its very self into existence through logic alone.[29]

In the pages following the above quote Penrose makes the following observations:

> …almost all the 'conventional' interpretations of quantum mechanics ultimately depend upon the presence of a 'perceiving being'…[30]

And:

> The issue of environmental decoherence … provides us with a merely stopgap position … 'lost in the environment' does not literally mean that it is *actually* lost, in an objective sense. But for the loss to be subjective, we are again thrown back on the issue 'subjectively perceived – by whom?' which returns us to the consciousness-observer question.[31]

And:

> …the behaviour of the seemingly objective world that is actually perceived depends on how one's consciousness threads its way through the myriads of quantum-superposed alternatives. In the absence of an adequate theory of conscious observers, the many-worlds interpretation must necessarily remain incomplete.[32]

And:

> As far as I can make out, the only interpretations that do *not* necessarily depend upon some notion of 'conscious observer' … require some fundamental change in the rules of quantum mechanics…[33]

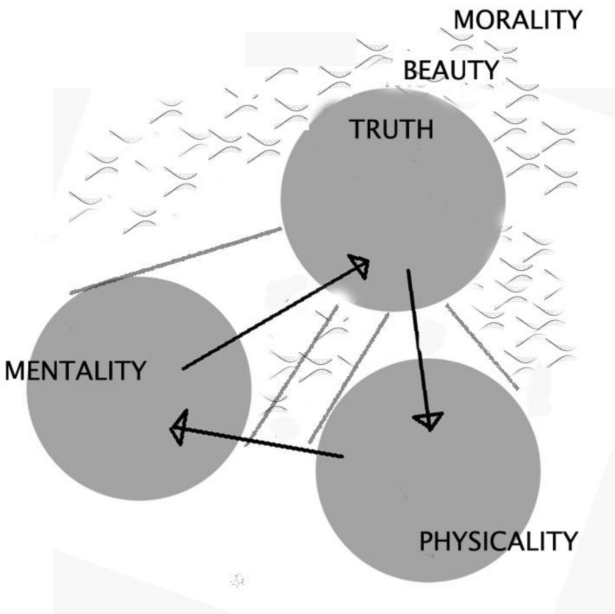

Fig 9 – Penrose's fig 34.1

Given these observations, and others in the same vein he makes elsewhere, the only thing that can possibly keep Penrose from coming to the most obvious conclusion that it is a some kind Universal Mindnature, not an insubstantial Platonic realm, that provides the ground of the process of the universe and therefore stands at the end of his road to reality (fig 10) is his distaste for the conclusion. Penrose, however, seems to equate a *non-physical* mind or consciousness with unreality:

> …I envisage that the phenomenon of consciousness- which I take to be a *real* physical process, arising 'out there' in the physical world.[34]

It seems, then, that Penrose considers that non-physical mind is far too unreal to generate itself! Something completely and absolutely different, such as mathematics, has a much better chance! But isn't mathematics itself a product of Mind? Is it not the product of the interaction of the subjective and objective aspects of universal Mindnature (a term which derives from Buddhist Dzogchen philosophy, it indicates a universal nondual primordial ground which is of the nature of mind), a nature which has been revealed in its objective manifestation by quantum physics? As Penrose says:

> If we are to believe that any one thing in the quantum formulism is 'actually' real, for a quantum system, then I think that it has to be the wavefunction …[35]

And it would seem to be the case that the evidence is stacking up in favor of the view that the nature of the level of the quantum wavefunction is primarily a Mindnature rather than 'physical' in the traditional sense. As the significant physicist Henry Stapp tells us:

> We live in an *idealike* world, not a matterlike world.' The material aspects are exhausted in certain mathematical properties, and these mathematical features can be understood just as well (and in fact better) as characteristics of an evolving idealike structure. There is, in fact, in the quantum universe no natural place for matter. This conclusion, curiously, is the exact reverse of the circum-stances that in the classical physical universe there was no natural place for mind.[36]

Indeed there are number of significant physicists moving towards a view that is consistent with Buddhist metaphysical perspectives that the ultimate nature of reality can only be described in terms of Mindnature (*Yogachara/Chittamatra* – Mind-Only, *Dzogchen* – Great Perfection)[37]. Such a move in our conception of reality, of course, requires a new understanding of the concept of 'physical', Stapp for instance has indicated that he now employs this term to indicate those aspects of reality which are measurable and he does not imply with its use the existence of a Cartesian type 'material' reality[38]. Many physicists and philosophers, however, seem to remain imprecise and confused in their use of the term.

In his paper *Nondual Quantum Duality* Stapp has indicated that quantum theory now requires us to conceive of reality as having a metaphysical structure which involves a deep Mind-like nondual ground which gives rise to the dualistic realm of experience:

> … in contrast to the application to classical mechanics, in which the physically described aspect is ontologically matterlike, not mindlike, in quantum mechanics

the physically described part is mindlike! So both parts of the quantum Cartesian duality are fundamentally mindlike. Thus quantum mechanics conforms at the *pragmatic/operational* level to the precepts of Cartesian duality, but reduces at a deep *ontological* level to a fundamentally mindlike nondual monism.[39]

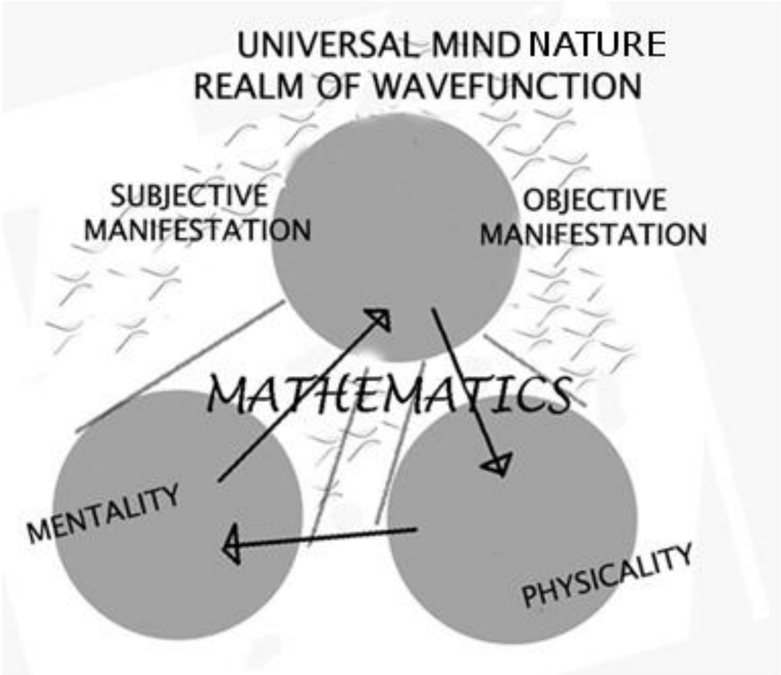

Fig 10

This view of the interdependent genesis of the two realms of dualistic manifestation; the realm of 'physicality', which is the objective aspect of the dualistic manifestation from the deeper, unitary, implicate (to use a term for levels of non-duality used by physicist David Bohm) dimension of Mindnature, and the subjective realm of individuated 'mentality' solves a crucial puzzle that has bothered many physicists and mathematicians. Eugene Wigner, for instance, referred to what he considered to be the 'unreasonable effectiveness' of mathematics in describing and explaining the physical world of 'nature'; he called mathematics a 'miracle' and 'a wonderful gift that we neither understand nor deserve.'[40] However, if both the realm of mentality and physicality emerge from a deeper level of universal Mindnature then it is surely not such a great mystery that mathematics, itself a product of mind, produces the conceptual patterns generated and followed by the 'physical' functioning of reality.

Penrose also refers to the 'mystery' of the 'remarkable relationship between mathematics and the actual behavior of the physical world.'[41] But, when it is realized that both are manifestations from a deeper realm of Mindnature, the 'mystery' disappears. The fact that individuated mind can, with a little thought on the matter, mirror its own deeper processes, the processes that underlie the *appearances* of 'matter' in the first place; is hardly a matter which should stretch the power of our minds; it seems quite natural when you put your mind to it!

Penrose seems desperate to avoid being forced against his better judgment, so to speak, to conclude from the weight of evidence that some kind of universal Mindnature or Consciousness underlies the process of reality and this leads him to suggest that somehow mathematics can somehow generate itself separately from mind or consciousness. 'Mathematics', he says 'is a kind of necessity, virtually conjuring its very self into existence through logic alone'[42], a formulation which seems to suggest that 'logic' is somehow capable of functioning all on its own, independently of Mind or minds, a suggestion which hardly seems plausible.

Implausible as it may seem, however, it is a view held by others, most notably Max Tegmark:

> I am a mathematical fundamentalist: I single out math as underlying the structure of the universe … I adopt the formalist definition of mathematics: it is the study of formal systems. Although this pursuit itself is of course secondary to the human mind, I believe that the mathematical structures that this process uncovers are 'out there', completely independently of the discoverer.[43]

In this quote we immediately can see the signs of a belief in the inherent existence of mathematical structures which are conceived of as somehow independent of Mind, even though Tegmark clearly recognizes that human minds are necessary to mediate them into the world of experience.

This is an example, albeit a very subtle one, of the kind of completely implausible picture of the meaningful somehow emerging from a complexity of meaningless units of mindlessness (mathematical or logical symbols are hardly meaningful without some minds being around). An amusing extreme materialist example of this is provided by Douglas Hofstadter's bizarre notion of a rudimentary mind arising from the machinations of a vast number of beer cans (he doesn't say whether they are full or empty):

> …beer can model of thinking or sensation … vast processes involving millions or billions or trillions of beer cans…[44]

It must be said at once, and with haste, however, that Penrose and Tegmark's depiction of the meaningful arising from the meaningless is much more, by far, sober and restrained than Hofstadter's unwitting parody of materialism, we would not wish the reader to think that they had been drinking at the same bar!

Hofstadter, however, does offer a more mathematically oriented suggestion for his thesis of the enforced meaningful meaninglessness arising through the operation of mindless complexity. In his 1970's smash hit book *Gödel, Escher, Bach: An Eternal Golden Braid*, under a section heading 'Meaningless Symbols Acquire Meaning Despite Themselves' Hofstadter informs us that:

> …a crucial part of my book's argument rests on the idea that meaning cannot be kept out of formal systems when sufficiently complex isomorphisms arise. Meaning comes in despite ones best efforts to keep symbols meaningless.[45]

This is Hofstadter's strange notion that mind and meaningfulness arise from 'strange loops' of meaningless symbols. Imagine the scenario: a strange loopy professor tosses a bunch of meaningless symbols into an empty room, hoping against hope that they will behave themselves for a change and just lay about meaninglessly on the floor where they land. But, as he fearfully suspects they might do, they immediately start vigorously arranging themselves into meaningful patterns once they notice that there are some 'sufficiently complex isomorphisms' between various sub-patterns that they meaninglessly fall into.

Dreading that some meaningful result might be the upshot of all this meaningless activity, the professor (probably of cognitive studies) dives into the room to make his 'best efforts' to ensure that the symbols remain meaningless. He battles meaningfully against the meaningless tide of mounting meaning, but to no avail and eventually gives up. He dejectedly walks away, realizing his mistake. The bunch of meaningless symbols he threw into the room actually comprised a 'formal system!' He therefore makes a determined resolution never to throw a meaningless bunch of symbols which make up a formal system into a room again! Does this appear to be a meaningless jumble of symbolic drivel masquerading as meaningful? Exactly! But it faithfully highlights the meaningless nature of Hofstadter's proposal. Formal systems must be meaningful to some extent in order to be formal systems in the first place, so the notion of struggling to keep them meaningless is, well, meaningless!

Once again it must be pointed out that Hofstadter's strange looped ideas represent an extreme example of the proposal that the meaningful can spontaneously erupt from pure meaninglessness, but, nevertheless, Penrose does seem to come close to some such proposal. Mathematics, he tells us, conjures 'its very self into existence through logic alone.' Compare this with Hofstadter's description of how he conceives of a 'Gödelian strange loop' being the origin of the generation of meaning from meaninglessness:

> …the Gödelian strange loop that arises in formal systems in mathematics (i.e. collections of rules for churning out an endless series of mathematical truths solely by mathematical symbol shunting without any regard to meanings or ideas in the shapes being manipulated) is a loop that allows such a system to 'perceive itself', to talk about itself, to become 'self-aware', and in a sense it would not be going to far to say that by virtue of having such a loop, a formal system acquires a self.[46]

This extreme presentation of the 'self'-generation of meaning from meaninglessness viewpoint clearly brings out the missing link in the proposed evolution of meaning from meaninglessness, the missing ingredient is, in fact, 'meaning'. The notion that 'mathematical truths' are generated, and also recognized, 'solely' by a meaningless 'mathematical symbol shunting' is simply incoherent for precisely because of the meaning of the term 'coherent' previously indicated concerning the meaning of the word 'coherent'. If two concepts are defined to be completely and absolutely devoid of significant connecting inherent qualities then to assert that one of them can arise from the other in any meaningful way is meaningless. The notion that absolutely meaningless symbols, existing in some fashion independently of, and unrelated to, human awareness, would have some kind of internal structure to make them a 'formal system' is a subtle form of materialism and, like all materialist explanations, it relies on an illicit investment to the units claimed to be meaningless with the meanings already invested by minds and Mind.

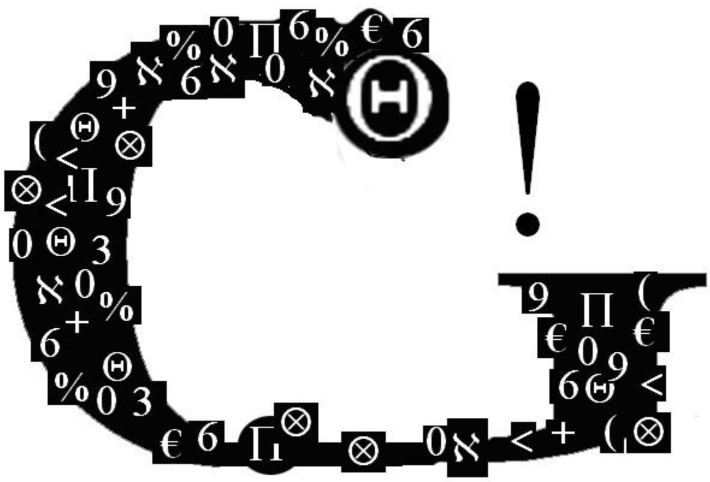

Fig 11 – Meaningless symbols perceiving themselves!

The Hofstadterian 'beer-can' or 'meaningless-symbols-only' model of meaning and perception illicitly, and deceptively, uses the reader's own intentionality to inject meaning into a putative meaningless conceptual image and then surreptitiously uses a judicious use of language to imply that the meaning is generated within the conceptual image itself. The notion of a '*strange* loop', for instance, primes the reader to expect something other-worldly to occur, and, of course, you need something 'other-worldly' to happen in order to produce meaning from the meaningless. We are told that a strange loop *is* a loop that 'allows' 'self-perception' and the ability of symbols to talk to them-selves (!) (fig 11) and so on, but nowhere in a one and a half thousand page book is there an account of exactly how this happens. The reader supplies all the meaning his or her self!

In his more recent book *I am a Strange Loop* Hofstadter uses lots of pretty colour pictures of hands trying to grasp computer generated images of self-referential mathematical whirlpools in order for his readers to get a better grasp of Hofstadter's loopy conception of what a 'self', and what consciousness is. In *Gödel, Escher, Bach: An Eternal Golden Braid* seems to suggest that even a book with enough internal self-referentiality would develop a rudimentary consciousness. Given the fact that *Gödel, Escher, Bach* itself must be the book to end all books for containing symbolic demonstrations of, allusions to, metaphors for, not to mention Escher pictures of, self-referentiality, it is truly amazing that all the volumes of this work do not get down of off bookshelves and begin composing further works of self-referentiality. Hofstadter's, and to a lesser extent Penrose's, viewpoint is a type of subtle materialism and therefore acts as an inverting distorting mirror in that it requires that meaningless and perception-less acts of self-perception (!) create meaning and perception, but how?

The evidence of the quantum 'self-collapse' of the wavefunction within the fundamental field of Mind, however, clearly indicates that the function, or capacity, for self-perception is a fundamental feature of the ground of reality and existence, and this vital spark of cognitive tendency, which Buddhist Dzogchen calls the 'primordial pristine cognitiveness', is internal to the fundamental quantum field of reality. Within both Yogachara-Chittamatra (Mind-Only) and Dzogchen (Great Perfection or Completion) Buddhist metaphysical perspectives the primordial ground is not only conceived of as a field of 'empty' potentiality (which is to say 'empty' of any particular manifestation), it is also asserted as have the fundamental and inseparable function of cognition. The ground of the universe is an infinite pool of potentiality and awareness, or empty-cognizance, which must create the infinite 'illusions' within the dualistic experiential realm because of its fundamental nature of awareness has the impetus to explore its own nature through cognitive activity. Herbert V, Guenther, in his book on Dzogchen metaphysics *The Matrix of Mystery* explains this 'pristine' cognitiveness of the fundamental 'matrix':

> What this term refers to derives directly from the self-excitatoriness (*rang-rig*) of the field as the universe of and for experience, and as such denotes a sensitivity and alertness that makes cognition possible as such on every level of the bio-sphere. This pristine cognition has a self-referential intentionality of atemporal primordiality…[47]

Here we are reminded of the highly regarded physicist John Wheeler's vision of the universe as a 'self-synthesized' universe:

> Directly opposite to the concept of universe as machine built on law is the vision of *a world self-synthesized.* On this view, the notes struck out on a piano by the observer participants of all times and all places, bits though they are in and by themselves, constitute the great wide world of space and time and things.[48]

This is the Dzogchen 'self-excitatory universe', which comes into being through an infinite web of internal self-perceptions. The only way that the universe could 'unfold' from within itself in this manner is if the fundamental quantum ground contained both the potentialities and the cognitive mechanism of perceptual 'unfoldment' within its own nature:

> In Dzogchen thought there is the additional factor of intelligence which inheres in the very dynamics of the universe itself, and which makes primordiality of experience of paramount importance. The atemporal onset of this unfoldment occasions the emergence of various intentional structures…[49]

And, of course, within this process lies the origin of the extraordinary manifestation of the meaning structures of mathematics. As this entire process of unfoldment is driven by an internal fundamental cognitive self-perception it should come as no surprise that that Gödel's theorems, which apply self-referentiality to the internal logic of mathematics, have metaphysical implications!

In the following few pages we are going to examine a formal operational method, which is to say a sequential imaginative procedure that has a distinct logical structure, which describes a sequence of steps by which the set of natural numbers - 0, 1, 2, 3, 4 … and so on, can be conceived of as 'coming into being' through acts of cognition. It was proposed by the nineteenth century mathematician George Cantor (based on the ideas of Giuseppe Peano). The sequence of steps begins with the idea of the 'empty set', which we can correlate with the 'empty' quantum field awaiting the first act of the creation operator, which in turn can be considered as a mathematical correlation of fundamental quantum cognition.

A 'set', which is an important mathematical idea, is a collection of objects, any objects; so the following is a set which contains an apple, an orange and a pear:

Fruit set = { apple, orange, pear }

A set, as shown, is contained in curly brackets '{ }' and the objects inside the brackets are called 'members'. So the following set:

Set of first 5 natural numbers = { 0, 1, 2 , 3, 4 }

has five members, all of which are natural numbers. This set therefore is said to be a subset of the complete set of natural numbers, which we will denote by N. The set of natural numbers is:

N = { 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, … }

This set is infinite, so it continues forever without end. Sets can contain other sets so the following set contains the set of natural numbers N and the set of the first 5 natural numbers:

Set containing 2 sets = { N, {0, 1, 2, 3, 4} }

Notice that the curly brackets which contain the set of the first 5 natural numbers are inside the curly brackets which contain the two contained members (the set of natural numbers and the set of the first 5 natural numbers).

The empty set contains no elements and is denoted by ∅ so:

$$\varnothing = \{\ \}$$

The following is a set that contains the empty set, which is not the same as the empty set:

Set containing empty set = { ∅ }

The following is a set that contains the empty set and the set of natural numbers:

Set containing empty set and set of natural numbers = { ∅ , N }

This is the set theory notation required in order to understand the following discussion.

We take the existence of the empty set as the starting point for our generation of the sequence of natural numbers; and we associate the empty set ∅ with the first natural number **0.** From this 'empty' beginning we can posit, which means to 'put or fix in place' or 'to postulate' or even 'cognize', the first set which actually contains something; and the 'thing' that this first containing set contains is the empty set! So the first containing set is associated with the natural number 1; we shall express this by saying that the set is denoted by 1. We have, therefore, as our first two sets with their associated numbers:

$\mathbf{0} = \varnothing = \{\ \}$

$1 = \{\varnothing\}$

Remember that ∅ is not the same as {∅}; the set ∅ has no members whereas the set {∅} has one member. The natural numbers associated with each set, therefore, are the number of members contained in the generated sets. The next set which gets generated, so to speak, contains the two set that have previously been generated:

$2 = \{\varnothing, \{\varnothing\}\}$

And this is how the cascading process of the generation of the natural numbers continues.

$3 = \{\varnothing, \{\varnothing\}, \{\varnothing, \{\varnothing\}\}\}$

$4 = \{\varnothing, \{\varnothing\}, \{\varnothing, \{\varnothing\}\}, \{\varnothing, \{\varnothing\}, \{\varnothing, \{\varnothing\}\}\}\}$

$5 = \{\varnothing, \{\varnothing\}, \{\varnothing, \{\varnothing\}\}, \{\varnothing, \{\varnothing\}, \{\varnothing, \{\varnothing\}\}\}, \{\varnothing, \{\varnothing\}, \{\varnothing, \{\varnothing\}\},$

$\{\varnothing, \{\varnothing\}, \{\varnothing, \{\varnothing\}\}\}\}\}$

As soon as the next set, and therefore the next natural number, is generated by the generation process, an immediately succeeding 'gathering' process takes over. This gathering process takes all the sets that have previously been 'generated' and then puts them into a new set. Once the new set has been gathered into a unity within the new set it is then posited, and thereby 'generated', as a new existent set entity. This creates a new existent entity which, in its turn, must be 'gathered' into the next set unity which is then posited. This continuous process of conceptual 'gathering' and 'positing' creates, in a conceptual lightening flash, the infinite field of natural numbers (fig 12).

At the moment this number generating lightening flash of cognitive-conceptual gathering and positing is purely formal; it is a conceptual creation of the logical structure of the kind of process which would have to take place in order for the natural numbers to be 'generated' from the starting point of the empty set, which in experiential terms we can think of, of course, as the mere awareness of emptiness, or the mere awareness of potentiality without content. Penrose says of this generative process of the existence of the natural numbers:

> …things like the natural numbers can be conjured literally out of nothing … this 'existence' can seemingly be conjured up by, and certainly accessed by, the mere exercise of our mental imaginations without any reference to the details of the nature of the physical universe.[50]

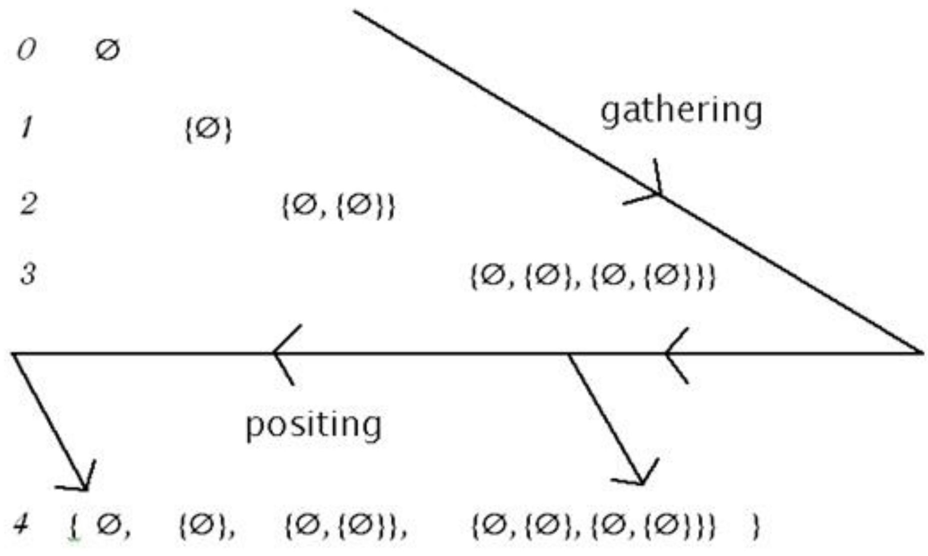

Fig 12 – Conceptual lightening flash of gathering and positing generating the natural numbers.

It is certainly both intriguing and remarkable that given the basic ground of these two logical aspects, the mere awareness of emptiness and the process of a spontaneous interdependent mechanism of the 'gathering and positing of the gathered', the entire infinite set of the natural numbers are spontaneously generated.

At the moment it might seem that Penrose is correct to allocate this process to 'the mere exercise of our mental imaginations', thereby not having much to do with 'the details of the nature of the physical universe.' But, on the other hand, the analysis that we have carried out above has clearly indicated that the mental-physical dichotomy cannot be, and indeed is not, absolute, but emerges from a deeper interconnected field of Mindnature. Furthermore we have established that Mindnature has the features of an empty field of potentiality within which there is an internal function of cognition, which is exactly what is required to generate through cognitive acts the natural numbers. Mindnature rather than logic, then, can quite naturally conjure the natural numbers from out of emptiness!

Probably the most significant, fundamental and mysterious features of mind and consciousness, or fundamental Mindnature, is exactly its capacity to be both unitary and yet encompass a vast multiplicity of diverse perceptions within the unity of its perceptual field. This is the gathering and positing function that produces multileveled conceptual systems of all kinds. Perception itself it a multileveled process, with lower-level percepts being gathered together into higher-level perceptions. The higher level perceptions, of course, are automatically 'posited' as 'existent' perceptions, for, if this were not the case, they would not be 'perceptions'. The very process of gathering and positing, therefore, actually lies at the heart of the process of perception and conception. And, furthermore, as the very nature of the ultimate ground of consciousness, which is Mindnature, has been found to be emptiness, which is no-thing but pure potential for experience, together with the inner function of cognition, we find that at the very ground of reality lies the potentially for the cascade of positing of one-ness, or I-ness, which creates the field of natural numbers! This takes place through internal acts of self-perception within the 'empty' field of reality.

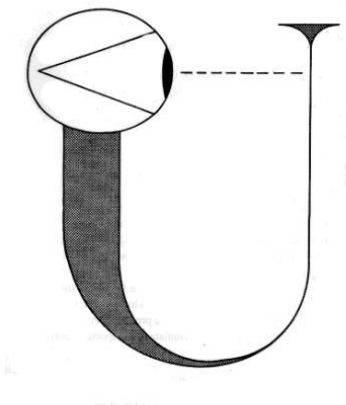

Fig 13

The physicist John Wheeler used his 'self-perceiving universe' image (fig 13) in order to illustrate his view that quantum theory necessarily indicated that the universe came into existence through multitudinous acts of self-perception, utilizing so to speak sentient beings as its cognitive agents. In my book *Quantum Buddhism: Dancing in Emptiness – Reality Revealed at the Interface of Quantum Physics and Buddhist Philosophy* I modified this image as shown in figure 14 to indicate how the fundamental ground of reality paradoxically explores its own essential nondual nature by producing a multitude of sentient beings, each with the illusion of their own personal **I**-ness. One might say that in the process of asserting its own existence the universe temporarily hides its essential unity; this occurs because the action of the internal cognitive impulse within the unified ground field necessarily disturbs the unity of the ground.

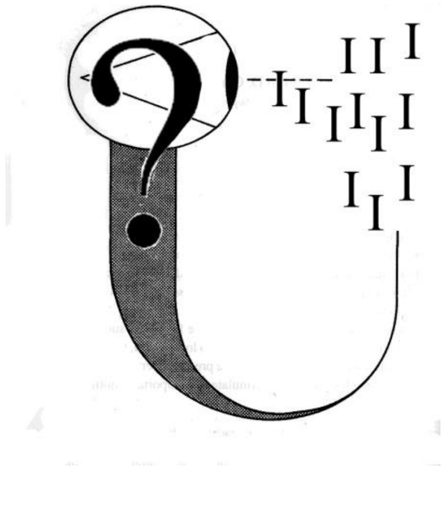

Fig 14

And it is a similar process of the cognitive assertion of unity, I-ness or one-ness, operating together with the unifying, or 'gathering' function of consciousness which generates awareness of natural numbers (fig 15).

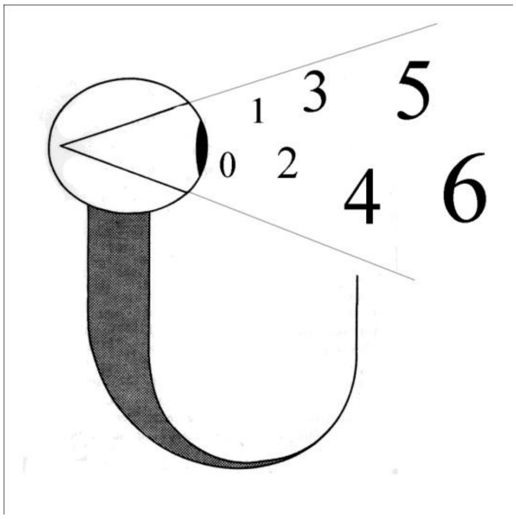

Fig 15

Fig 16 schematically shows the actual perceptual acts, movements of consciousness which simultaneously 'gather together' previous acts into a unity and then 'posit' the new unity as a new act, thereby naturally establishing successive natural numbers.

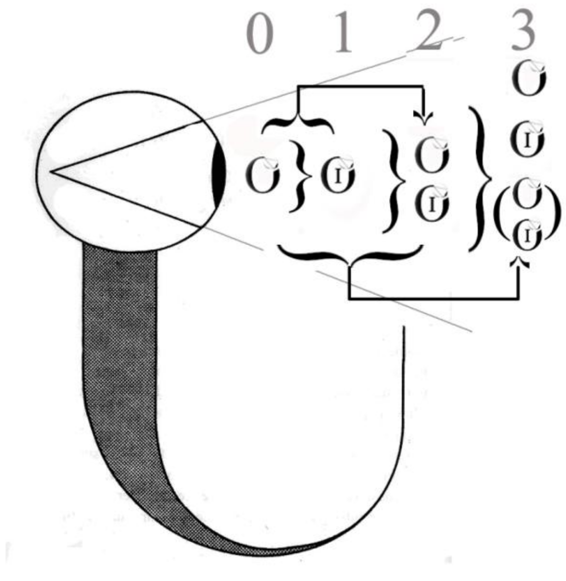

Fig 16

The extraordinary Thai meditation master Ajahn Chah, in one of his extemporary dharma talks, inspirational and glittering with crystal insights that he was widely famous for, spoke of the necessity for the understanding processes of consciousness; to actually be able to watch, and then deconstruct them within direct focused mediation practice in order to become aware of the fact that they were autonomous, natural and therefore not-'self':

> Whatever we experience, it all arises within this knowing. If this mind did not exist, the knowing would not exist either. All this is phenomena of the mind. … the mind is merely the mind. It's not a being, a person, a self, or yourself. Its neither us nor them. …The natural process is not oneself. It does not belong to us or to anyone else. It's not any thing.[51]

And:

> This mind is free, brilliantly radiant, and unentangled with any problems or issues… In the beginning what was there? There is truly nothing there. It doesn't arise with conditioned things, and it doesn't die with them.[52]

The basic field of the mere mind, which is just the vibrant, empty capacity for the fundamental act of knowing, provides the ground from which all the phenomena of the experiential dualistic world emerges. It is just this fundamental Mindnature, the ground of knowing, so to speak, that provides the basis of both the coordinated appearances of the apparently external material world and the apparently 'internal' conceptual structures of mind by which the functioning of appearances are comprehended.

The entire vast array of appearances, experiences, reflective conceptual systems, and so on arise from a primordial flickering, knowing movement of consciousness within Mindnature that disturbs its quintessential unity:

> Please clearly understand that when the mind is still it's in its natural, unadulterated state. As soon as the mind moves, it becomes conditioned. … The desire to move here and there arises from conditioning. If our awareness doesn't keep pace with these mental proliferations as they occur, the mind will chase after them and be conditioned by them. Whenever the mind moves, at that moment, it becomes a conventional reality.[53]

And the most fundamental impetus which underlies the movement of mind is the perception of inherent existence of selfhood which cause the imputation of an '**I'** or a **'1'** into the unconditioned field of consciousness. This fundamental grasping at existence, which, paradoxically, appears to be an inner tendency within consciousness itself, must create the field of natural numbers through a natural cascade of 'inner' perception.

The ground which Ajahn Chah refers to as 'mere unconditioned mind' the physicist David Bohm described in the following terms:

> So we see that that the ground of intelligence must be in the undetermined and unknown flux, that is also the ground of all definable forms of matter. Intelligence

> is thus not deducible or explainable on any basis of knowledge (e.g. physics or biology). Its origin is deeper and more inward than any knowable order that could describe it.[54]

This fundamental ground of intelligence, which as we have seen Bohm clearly identified with the realm of the wavefunction, constitutes the very source of all intelligence, meaning and experience, it therefore cannot be explained by anything other; it is the ground for all explanations.

This metaphysical perspective suggests that the perception and comprehension of the field of all number, as well as the mathematical cohesive conceptual structures which can be comprehended within this fertile ground of pure non-sensory meaning, arise from a basic ground of empty luminosity, or pure potential cognition. In this way the extraordinary variety and multiplicity for inwardly experienced mathematical meanings can be understood as developing out of the basis of the tendency towards self-perception within a fundamental ground of pure indeterminate meaning. The way in which we can conceive of this process generating the field of integers, which encompass positive and negative natural numbers, by a resonant, co-ordinated, amplificatory, perceptual process cascading from a tiny movement within an empty seed of potentiality is shown in fig 17. The empty seed is *sunya*, the zero point, the cosmic seed of emptiness which is 'swollen' with potentiality. One meaning of *sunya*, which is the Indian origin of the concept of zero, is 'the swollen', in the sense of an egg of potentiality which is about to burst into manifestation.

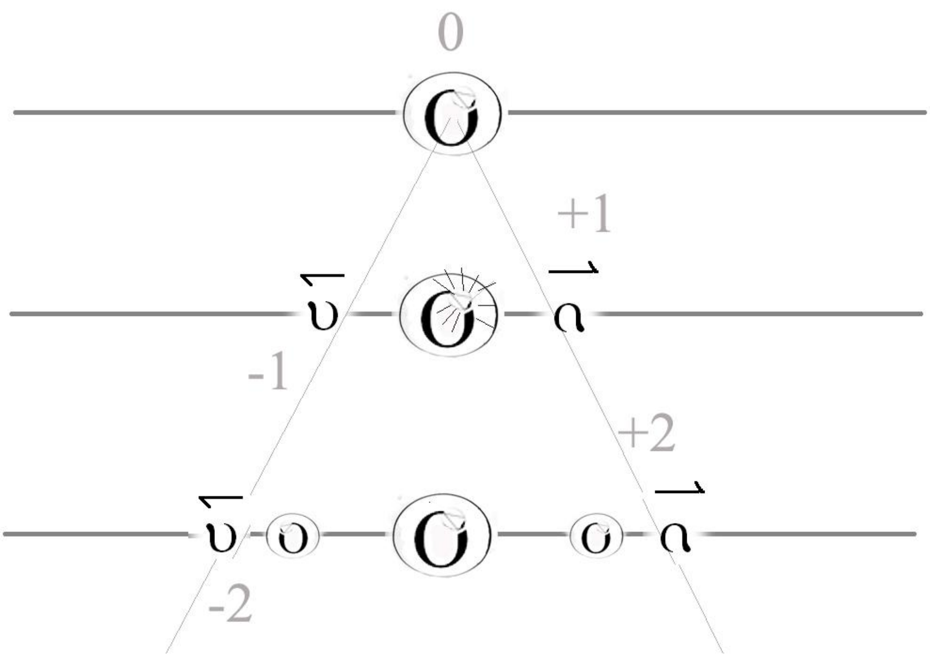

Fig 17

The coiled eye inside the egg represents the inner function of cognition which is part of the fundamental nature of the fundamental ground of consciousness. When the strength of the perceptual tendency reaches a certain level there is an inner pulse of perception within the non-dual, interconnected field of consciousness. This causes an interconnected disturbance within the field, positive on one side, negative on the other, these two balance each other.

Although these processes can be presented as purely logical structures which were considered by mathematicians such as Cantor and Gödel, like Penrose, to be residents of some kind of pure mathematical realm, the process of both the appearance of the 'physical' world and the apparent logical-mathematical realm of pure non-sensuous meaning which 'appears' to be directly accessible to individual consciousnesses (of sufficient subtly and training) becomes completely comprehensible when seen as the result of a deep resonant perceptual operation within a field of awareness-consciousness. The structures of both the external apparently independent material world, and the internal mathematical realm of meaning, would develop at a fundamental level of consciousness much deeper than the individual. It would follow, therefore, that the inner mathematical realm of pure non-sensuous meaning structures would appear to be as independent of the individual mind as the appearance of the material sensuous world. Mathematics, therefore, would appear to the individual mathematicians involved to be derived from an independent 'Platonic' realm of pure meaning.

The analysis of the process by which the natural numbers, and the positive and negative integers:

$$\mathbf{I = \{ \dots -3, -2, -1, 0, 1, 2, 3 \dots \}}$$

are generated injects a discrete numerical structure onto a fundamentally continuous field, and this becomes apparent as the whole numbers prove not to be sufficient thus giving rise to the use of rational numbers which are constructed by forming ratios, or fractions, of the form:

$$\mathbf{M/N}$$

Examples are, of course: ½ , ¾ , 7/8, 4/15 , 678/10254. All this was known to the Greek world, indeed the Pythagoreans made a religion of their appreciation of the power of number in the task of understanding the structure and process of reality. However, as we have seen, when the awful mathematical truth of the existence of irrational numbers was discovered the numerical religious faith of the Pythagoreans was sorely tested. The existence of the irrational numbers was the first ominous sign of the ultimate emptiness of mathematics, which was demonstrated by Gödel.

We have already employed the philosopher Alain Badiou's metaphor of the 'swarming' of numbers and it is now necessary to look into this dense numerical multitude in a little more detail. Consider the number line, as the imaginative geometric representation of all possible 'real' numbers is called, between the natural numbers 0 and 1. We might note in passing that the term 'real' in this context was employed in order to reassure the frayed nerves of mathematicians in the face of the irrational numbers; giving the wayward entities the accolade of inherent reality was probably thought to bring their irrationality within acceptable conceptual limits, so to speak. If we take any two rational numbers within this range, 1/3 and ½ for example, then it is always possible to construct another rational number between them by finding the mid point. This is most easily visualized by expressing the numbers in terms of a denominator which is twice their common denominator. In the case of the denominators 2 and 3 the common denominator is 6 (multiply the denominators together) so twice that gives 12:

$$1/3 = 4/12$$

$$1/2 = 6/12$$

The mid point between these two rational numbers can be easily seen to be 5/12 (fig 18). This process of taking the midpoint can be repeated endlessly, or infinitely, as shown. This means that, without even considering the irrational numbers, there must be infinite number of rational numbers within any segment of the number line.

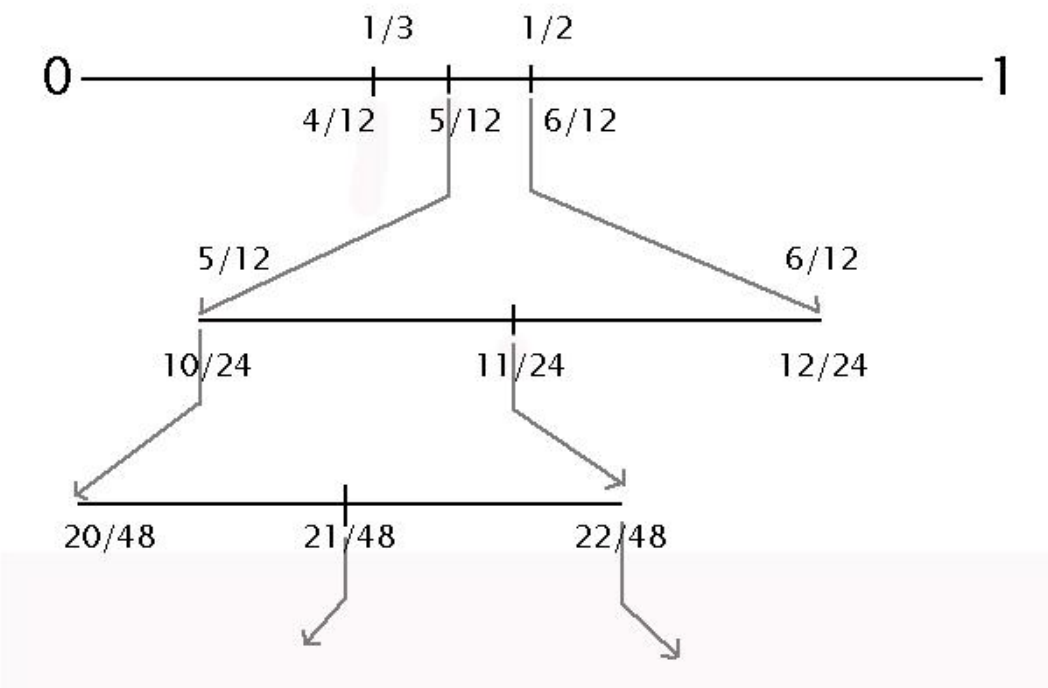

Fig 18

When we count in the irrational numbers the situation becomes even more infinitely infinite! And this characterization is precisely correct because the infinite size of the infinity of the irrational numbers is actually larger than the infinite size of the infinity of the rational numbers. Because of this fact it turns out that there are more real numbers, which are the rational and irrational numbers taken together, in the gap between zero and one than there are natural numbers altogether. The mathematicians Robert and Ellen Kaplan remark, concerning this, that:

> … we have just found a second and larger size of infinity (and the hairs on the back of the neck stand up at the hint of perhaps more). It is hard to think of a comparable shock to the life on the mind (unless it be the revelation that others think ‘I’)[55]

This situation clearly indicates a hierarchy of infinities each nested inside the one higher; and also each one contained within the one which is deeper). As Kaplan and Kaplan write:

> The infinite disguised as the indefinite is our … begetter. But in this same guise it is how we imagine the world truly to be: made up ultimately not of separate objects, molecules, atoms, electrons or quanta, but, past the ever more granular, to be as partless as the ocean, where our little prisms of selves spray up and soon enough submerge.[56]

We find an apparently inherently existent structure of independent entities, the natural numbers, the integers and the real numbers, disappearing into deeper realms, through the irrational and transcendental numbers, and ultimately into the formless realm of potentiality, the empty realm of sunyata, the 'swollen' zero potentiality which lies at the heart of reality. Robert and Ellen Kaplan identify this ground of meaning and experience with the Greek concept of the *apeiron* which means 'without boundary'. According to Anaximander who lived a hundred and fifty years before Socrates, '…the source is the *apeiron* – as if distinction rose out of indistinction.'[57]

Despite the indications that the foundations of mathematics might not be as robustly 'logically' solid as often hoped, the term 'logical' being taken as implying the absolutely independent existence of ultimate logical entities and procedures, attempts had been made to demonstrate such a pristine, logical scaffolding for mathematics. The work of Kurt Gödel, however, negated the logical possibility of such aspirations. Gödel's theorem is accepted by the community of logicians, mathematicians and philosophers as being of the foremost importance within the foundations of logic and mathematics. The full extent its significance, however, is a matter of some controversy. Penrose stands at the head of those who believe that Gödel's theorem opens out insights into the nature of mind:

> It was in 1930 that the brilliant young mathematician Kurt Gödel startled a group of the world's leading mathematicians and logicians, at a meeting in Königsberg, with what was to become his famous theorem. It rapidly became accepted as being a fundamental contribution to the foundations of mathematics-probably the most fundamental ever to be found-but I shall be arguing that in establishing his theorem, he also initiated a major step forward in the philosophy of mind.[58]

As we shall see Penrose is correct in his evaluation; for Gödel's theorem is exactly what we would expect of a self-perceiving universe!

The actual details of the proof are too complex to be presented here, and it is not necessary to have a full comprehension of them. The actual result and the deep philosophical implications can be adequately presented without a great deal of difficult logic. What Gödel did was to prove that there are true arithmetical propositions that are not provable; a remarkable and strange achievement which required the use of a mathematical strange loop, a self-referential paradox.

A usual prologue to an explanation of Gödel's use self-referential paradox is the presentation of the liar's paradox:

This very sentence is false.

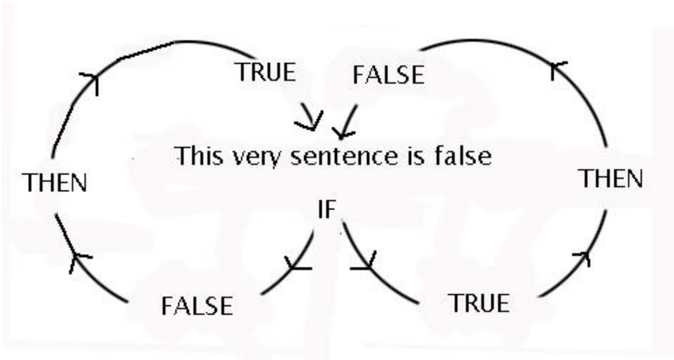

Fig 19

Considering the truth possibilities of this innocuous looking sentence sends the reader on an infinite loop of hovering between the extremes of truth and falsity; as indicated by fig 19. If we take the right hand path and assert that the sentence is true then the actual assertion of the sentence makes the sentence false, this forces us now down the left hand path wherein the sentence is false, this assertion of falsehood, together with the sentence's own assertion of its falsity, renders the sentence true, and so on.

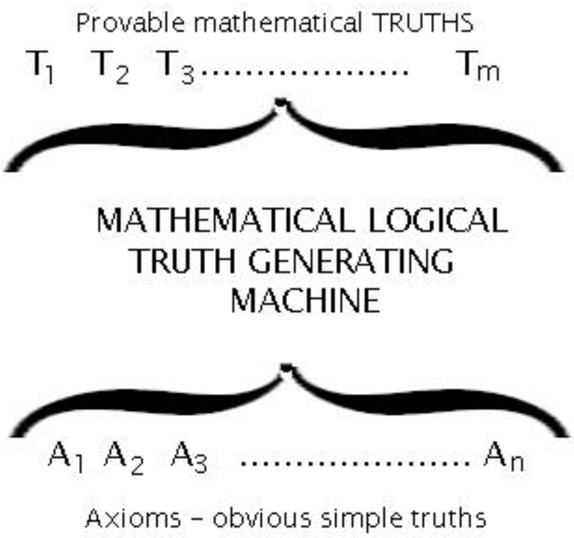

Fig 20

The view of mathematics that was fundamental to most mathematicians own understanding of the nature of their discipline prior to Gödel, is indicated by fig 20. At the bottom the basic fundamental axiomatic truths, which can be seen to be true by their own internal logical nature, so to speak, are fed into the flawless, pre-existing, although not fully discovered, mathematical truth generating machine. By using the mechanisms of the mathematical truth generating machine new, more complicated and illuminating mathematical truths can be generated, or proved. On this view it is the basic task of mathematicians to discover all of the details of the mechanisms of the mathematical truth machine so that all of the truths which can be generated are generated. There are two extremely important features that this 'formal system' of the axioms and the generating machine *must* contain:

> **Completeness** – the system generates all possible truths that are contained within the axioms.
>
> **Consistency –** the system must not contain any hidden contradictions. Any logical contradiction, or inconsistency, within the system renders it useless Anything can be proved on the basis of an inconsistent system.

In order to demonstrate the complete logical inviolability of the mathematical edifice, therefore, it was necessary to show, beyond any possible doubt, both consistency and completeness. Gödel, showed the opposite (fig 21).

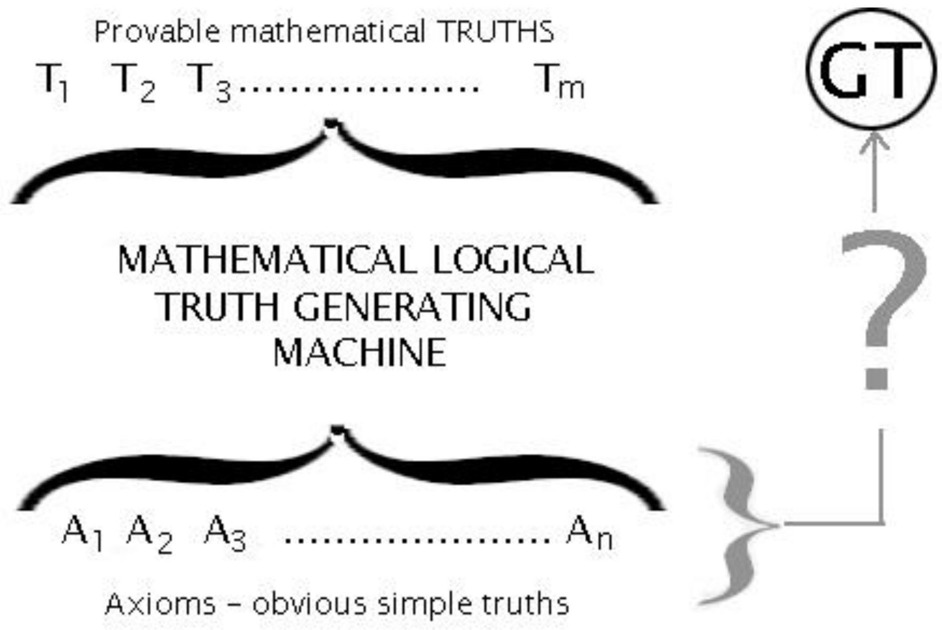

Fig 21

GT is the Gödel truth, or statement, which is known to be true but cannot be generated, or proved, by the mathematical logical generating machine. The Gödel statement is:

This very statement is not provable within this system.

The crucial issue is that of how Gödel was able to derive a statement, within the formal logical system of arithmetic, that can be known to be true at the same time as it asserts its own unprovability? This seems like self-referentiality gone mad! If the statement is unprovable, by its own admission so to speak, how do we know that it is true? It is this feature of the theorem wherein lies the spectacular logical genius of Kurt Gödel. The logical trick that Gödel used was a coding of the symbols which are used within logical proofs into numerical values. This was his famous, at least in mathematical circles, Gödel numbering system. The system used Gödel employed the fundamental fact that all numbers can be decomposed into prime factor. For instance the number 18900 can be expressed as the prime decomposition:

$$18900 = 2^2 \text{ x } 3^3 \text{ x } 5^2 \text{ x } 7^1 \qquad (= 2\text{x}2 \text{ x } 3\text{x}3\text{x}3 \text{ x } 5\text{x}5 \text{ x } 7\ )$$

This means that any sequence of natural numbers, in this case 2, 3, 2, 1, can be translated into a unique number, in this case 18900).

The following table shows a simple method of translating some logical symbols into numerical values:

Logical sign	**Gödel Number**	**Meaning**
z	1	variable (unknown number)
=	2	equals
‘	3	prime

Using this coding table the logical snippet ‘z=z’ can be coded as shown:

$$\text{z=z' - coded} \rightarrow 2^1 \text{ x } 3^2 \text{ x } 5^1 \text{ x } 7^3$$

$$= 2 \text{ x } 9 \text{ x } 5 \text{ x } 343$$

$$= 30870$$

In this way any string of logical symbols which make up a logical deduction can be coded as a, generally very large, unique number. The extraordinary system of transforming strings of logical symbols into numerical values that Gödel developed enabled him to test whether a logical proposition was provable or not by examining the nature of the number it translated into. And by using this incredible dual layer of meaning Gödel was able to rigorously construct a special Gödel statement and associated number that he could demonstrate, by its numerical value, to be unprovable. This number corresponded to a logical proposition that actually stated that it was a logical proposition that could not be proved! The subtle, and amazing, point in this procedure is that we know that the statement that the logical proposition is unprovable *is true* precisely because *the value of the Gödel number of the logical proposition guarantees its validity within the formal system*.

So now Gödel, and therefore we, know that there is a true statement within the formal system of arithmetic which states that:

This very statement is not provable within this system.

Hence the derivation of the Gödel theorem is able to prove, by using a trick to seemingly go outside the formal system, that this 'unprovable' statement is true! This establishes the fact that the logical system which underlies arithmetic is incomplete, which is to say there are truths within the system which cannot be proved.

It should be noticed that the Gödel statement is peculiarly self-referentially potent in that if it could be proved it would thereby be disproved; its proof would at the same time its disproof! This means that if the system was able to prove this statement then the system would have demonstrated the truth and falsity of same statement, a situation which meant that the system was inconsistent. This further means that if the system is consistent then the Gödel statement must be true; this means that *the consistency of the system depends upon the fact that there is at least one unprovable truth*. This is **Gödel's First Incompleteness Theorem**:

> Any consistent, which means capable of effectively determining truth and falsity, formal system that is complex enough to contain arithmetic, must contain unprovable truths, which means it is incomplete.

And this has a further correlative formulation which is **Gödel's Second Incompleteness Theorem:**

> Any consistent, which means capable of effectively determining truth and falsity, formal system that is complex enough to contain arithmetic, cannot prove its own consistency.

And the upshot of these remarkable theorems, or perhaps meta-theorems is a better description, concerning the formal nature of mathematical systems is that it is impossible for a formal system to validate itself. A complete and universal comprehension of the validity of the system requires a kind of meta-perception, an intuitive perception outside of the formal system itself. And in particular this intuitive requirement is especially essential in the domain of the infinite:

> The mathematician's intuitions of infinity-in particular, the infinite structure of the natural numbers-can no more be reduced to finitely formal systems than they can be expunged from mathematics.[59]

According to the eighteenth century philosopher David Hume:

> The capacity of the mind is not infinite, consequently no idea of extension or duration consists of an infinite number of parts or inferior ideas, but of finite number, and these simple and indivisible…[60]

But Hume, of course, simply did not have all the evidence required to decide upon this issue. In particular he certainly had no experience of meditative states of direct insight into the nature of consciousness which suggest that there is an ultimate experiential non-dual ground of reality, which is designated in the following as the 'basic space':

> Basic space and awareness are innately all-encompassing. Basic space is the absence of mental constructs, recognizing the complete emptiness of mind essence. Space and awareness are inherently indivisible.[61]

Furthermore this basic space, which is also designated as emptiness, or 'empty cognizance':

> …is totally beyond any kind of pigeonholing anything in anyway whatsoever. It is to be utterly open, beyond categories, limitations, and the confines of assumptions and belief.[62]

This basic space, the fundamental ground of consciousness, meaning and perception, therefore, corresponds to the Greek concept of the *apeiron*, the boundless or formless potentiality from which the forms of the dualistic realm, including numbers, emerge.

It seems quite clear, then, that Gödel' famous theorems point to the fact that there is a realm of direct knowing and understanding which transcends and underlies any particular forms of knowledge and understanding. In an interview Penrose elucidates this insight as follows:

> Basically the thrust of my argument is that the quality of "understanding" is something outside the capabilities of a computer … The generality of Gödel's argument simply illustrates how powerful conscious reasoning (through understanding) can be. Just following rules (which is what computers do—-albeit extraordinarily well) is something very different from understanding. (This is something that educationalists know very well!) I argue that understanding (whatever it is) requires "consciousness" (whatever "that" is!).
>
> To take the argument further, I take the view that the quality of consciousness is something that is potentially out there in the physical world, and is not necessarily something unique to human beings. But I regard the Gödel argument as showing that conscious understanding is something that cannot be properly imitated by a computer. So I argue that if consciousness is part of physics—-describable by the "true" laws of physics—-then the true laws of physics must be non-computable. It is known (using Gödel-Turing-type arguments) that there are many areas of mathematics which are actually non-computable, so I am claiming that the true laws of physics (not yet fully known to us) must also be non-computable. But the known laws of physics are (more-or-less) computable, so we must look outside the known laws. I argue, further, that the only plausible loophole in the laws that we know lies in the issue of quantum measurement, and that the "measurement paradox" (basically "Schrödinger's cat") points to where we need to make further progress in our understanding of the laws of physics in order to uncover what is actually non-computable in the true laws).[63]

In true Penrose style he goes straight to the heart of the matter, or rather to the heart of the lack of matter in his assertion that consciousness must be a central and ubiquitous aspect of reality and that the quality of understanding or knowing which is function of consciousness transcends and contains logical mechanism. However he still seems to cling to the hope that are inherently existent and finally discoverable 'laws of physics' which will bring a final comprehensive, and presumably logical in some fashion, understanding of the process of reality. But isn't it the case that 'the only plausible loophole in the laws that we know lies in the issue of quantum measurement, and that the "measurement paradox"' indicates, like Gödel's results, that all aspects of reality have their source in the fundamental 'empty cognizance' of Mindnature which lies beyond the capture of all conceptual systems. As the Zen master Hung Po explains a core Buddhist perspective:

> This pure Mind, the source of everything, shines forever and on all with the brilliance of its own perfection. But the people of the world do not awake to it, regarding only that which sees, hears, feels and knows as mind. Blinded by their own sight, hearing, feeling and knowing, they do not perceive the spiritual brilliance of the source substance. If they would only eliminate all conceptual thought in a flash, that source substance would manifest itself like the sun ascending through the void and illuminating the whole universe without hindrance or bounds.[64]

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[1] Penrose - Road to Reality p9
[2] Penrose - Road to Reality p12
[3] fq(x) Wigner's Gift Horse - Feb 1 2008
[4] The Taming of the Infinite p244
[5] Infinity p34
[6] Taming p23
[7] Number: The language of science p103
[8] Number p105
[9] Vedral, Vlatko (2010) p200
[10] A elucidation of the philosophical procedures with examples of the Madhyamaka will be published in a forthcoming issue of the journal in an article entitled Quantum Madhyamaka: The Illusion-like Nature of Reality.
[11] Infinity p3
[12] Infinity p61-62
[13] Infinity p69
[14] Infinity p3
[15] Alain Badiou
[16] Web Dictionary
[17] Roads to Reality p54
[18] Infinity p118
[19] The Analyst – section 75.
[20] BI p25
[21] Brunnhölzl, Karl (2004) p228
[22] Heart Sutra
[23] BI p32
[24] Victor Mansfield
[25] Emperor's New Mind, shadows of Realty, The Road to Reality
[26] Penrose - Road to Reality p22
[27] Penrose - Road to Reality p19
[28] Karmapa' Middle Way p207

[29] Penrose - Road to Reality p1029
[30] Penrose - Road to Reality p1031
[31] ibid
[32] ibid
[33] Penrose - Road to Reality p1032
[34] ibid
[35] Penrose, Roger (2005) p508
[36] Stapp, Henry (2004) p223
[37] This phenomenon will be covered in a forthcoming article 'Quantum Dzogchen: Nature and its Place in Consciousness'.
[38] Stapp ???
[39] Stapp - Nondual Quantum Duality
[40] fq(x) Wigner's Gift Horse - Feb 1 2008
[41] Penrose - Road to Reality p21
[42] See above quote from Roads
[43] Tegmark in On Math, Matter and Mind
[44] I am a Strange Loop p30
[45] GEB xxi
[46] ibid
[47] Guenther, Herbert V. (1984). p 24
[48] [48] Barrow, John D., Davies, Paul C. W., Harper, Charles L. (eds) (2004) p577 – Wheeler, J A (1999) 'Information, physics, quantum: the search for links.' In *Feynman and Computation: Exploring the Limits of Computers*, ed A. J. G. Hey, p309 (314). Cambridge, MA: Perseus Books.
[49] Guenther, Herbert V. (1984). p 38
[50] Penrose - Road to Reality p64-65
[51] Food For the Heart p181
[52] Food For the Heart p183
[53] Food For the Heart p179
[54] Wholeness and the Implicate Order p67
[55] The Art of the Infinite p240
[56] The Art of the Infinite p75
[57] ibid
[58] Shadows of the Mind p64
[59] Incompleteness p186
[60] A Treatise on Human Nature
[61] Dzogchen Primer p28
[62] ibid
[63] Shaking Up Foundations Of Math: Roger Penrose On Kurt Gödel's Groundbreaking Work
[64] Addiss, Stephen; Lombardo, Stanley; Roitman, Judith (2008) – Zen Sourcebook p39