

Article

The 'Core' Concept and the Mathematical Mind: Part II

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ABSTRACT

Pure mathematics is often seen as an 'inverted pyramid', in which algebra and analysis stand at the focal point, without which students could not possibly have a firm grounding for graduate studies. This paper examines a variety of evidence from brain studies of mathematical cognition, from mathematics in early child development, from studies of the gatherer-hunter mind, from a variety of puzzles, games and other human activities, from theories emerging from physical cosmology, and from burgeoning mathematical resources on the internet that suggest, to the contrary, that mathematics is a cultural language more akin to a maze than a focally-based hierarchy; that topology, geometry and dynamics are fundamental to the human mathematical mind; and that an exclusive focus on algebra and analysis may rather explain an increasing rift between modern mathematics and the 'real world' of the human population.

Part II of this article includes:

- 4: Puzzles and Games as an Expression of Human Mathematical Imagination
- 5: State Space Graphs and Strategic Topologies
- 6: The Brain's Eye View of Mathematics
- 7: The Fractal Topology of Cosmology

Key Words: core concept, mathematical mind, brain, real world.

5: State Space Graphs and Strategic Topologies

Virtually every puzzle, whether logical, conceptual, arithmetic, geometric, topological or strategic is navigated by a human subject in an abstract journey from beginning state to solution, through many possible cul-de-sacs in a journey which takes the form of a connected path along the nodes of a graph of states which constitutes a maze of intermediate positions. This is a process akin to a journey through the wilderness in which various conceptual attributes essential for solving the puzzle can point the way to the solution much as topographical signposts or at least sensibly reduce the huge space of possibilities to a feasible number of options.

Although every solvable puzzle is path connected, the form and size of the state graphs can vary extremely. A regular graph with a standard set of moves, such as the Rubik revenge cube, can have a huge state space. By contrast state spaces in which the transitions are complex, irregular

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may have a much smaller state space, despite being of non-trivial difficulty. We now examine several different types of puzzle to investigate the common topological thread involved in navigating a connected path from starting point to solution.

Example 5.1: Who Own the Zebra?

This logical puzzle sometimes incorrectly attributed to Einstein consists of a series of logical statements associated with five colours of house, five nationalities, five drinks, five pets and five brands of cigarette. The solution to the puzzle is most easily performed by making a table of the items, and then analyzing the logical statements, to specify successive entries of the table, branching to deal with contingencies as little as possible.

Given the statements listed below, we are asked: "Who owns the zebra?" and "Who drinks water?"

1. There are five houses.
2. The Englishman lives in the red house.
3. The Spaniard owns the dog.
4. Coffee is drunk in the green house.
5. The Ukrainian drinks tea.
6. The green house is immediately to the right of the indigo house.
7. The Old Gold smoker owns snails.
8. Kools are smoked in the yellow house.
9. Milk is drunk in the middle house.
10. The Norwegian lives in the first house.
11. The man who smokes Chesterfields lives in the house next to the man with the fox.
12. Kools are smoked in the house next to the house where the horse is kept.
13. The Lucky Strike smoker drinks orange juice.
14. The Japanese smokes Parliaments.
15. The Norwegian lives next to the blue house.

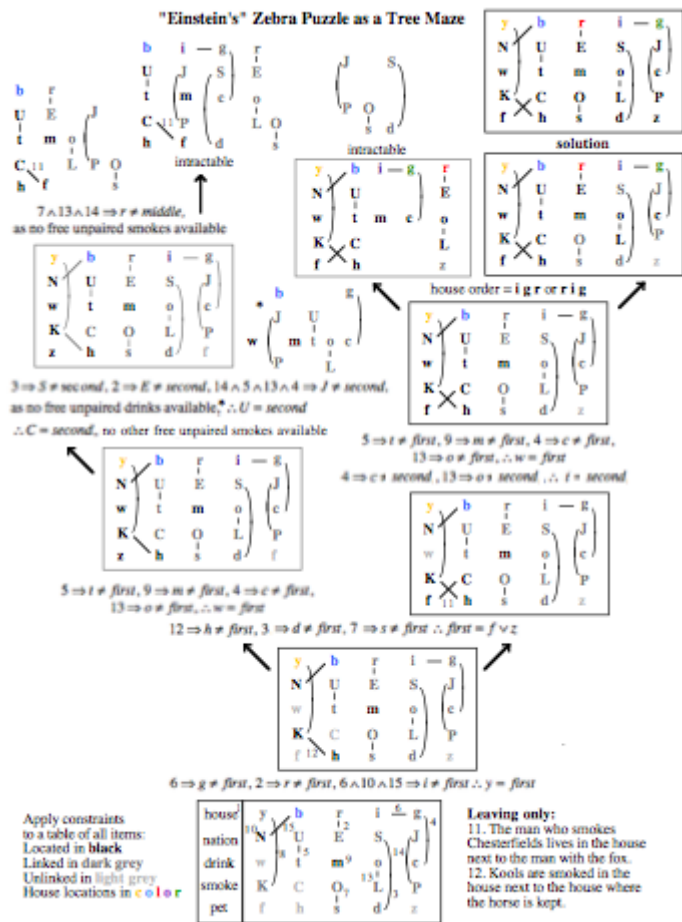
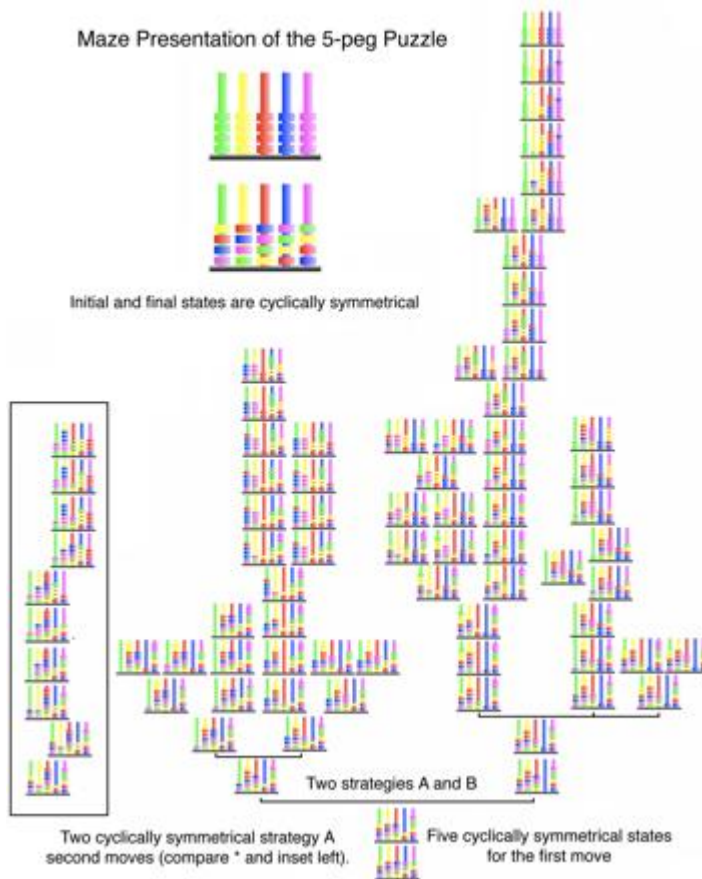


Figure 5.1: "Who Owns the Zebra?" portrayed as a strategic maze of puzzle states.

Figure 5.1 shows a decision-making tree maze for the puzzle, which is conveniently tabulated in the same way as the solution for ease of reading. Initially all but two of the statements are processed and incorporated into the table in terms of links between categories which determine the relative positions of the linked items. Although the puzzle is non-trivial its decision making



state graph is a tree with only a few nodes once the logical statements, which can be processed simultaneously are grouped into one step.

Because there are many ways of prioritizing the statements and in which order to deal with the categories, a human subject will frequently navigate a version of the tree, adding one or two extra assumptions, only to find they have reached an intractable position, returning to the trunk of the tree, or a variant of it, to try again, releasing some or all of the assumptions which led to intractability. In doing so, they are navigating a logical space, abstractly akin to making a path-connected journey in a topographical landscape.

Figure 5.2: Maze Presentation of example 5.2

Example 5.2 The Five Peg Puzzle

Figure 5.2 shows the maze for a puzzle in which one or two top rings can be moved, but only on to a ring, or empty peg of the same colour as the lower one. Again there are only a limited number of states because many moves rapidly lead to intractability. The graph now has trivial loops but is unidirectional upward because the moves are not reversible. Again the subject is traversing a conceptual territory, which can be described as a path-connected region.

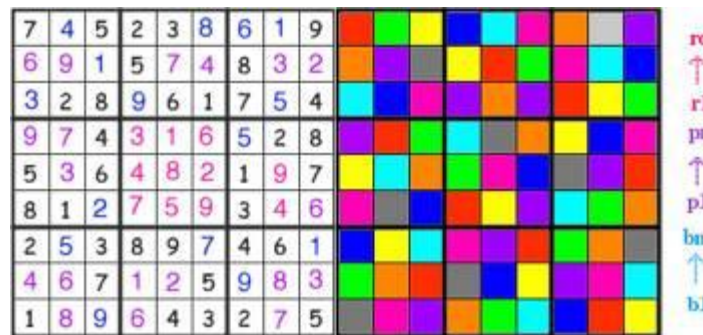


Figure 5.4 Elementary Sudoku (left) has no numeric operations, being based only on each row, column and sub-square having distinct entries, as the colour-coded version (centre) shows. Black: initial puzzle. Blue: clues from horizontal and vertical lines. Purple: clues using sub-squares. Red: final solution. This puzzle can be solved without contingencies and thus has a state space consisting of a meander maze - the unique path from start to solution (right). Similar colour codings are used to depict Cayley tables^[24], which also have distinct entries in every row and column.

Example 5.4 Elementary Sudoku

Elementary Sudoku illustrates the ultimate simplicity of state space structure. Although it presents as a numeracy puzzle, it is simply a category-matching puzzle, as illustrated by colour coding, requiring only that every row, column and sub-square has nine distinct entries. Because all the numbers can be found simply by filling in numbers determined in sequence from the provided clues, the state graph is just a line, as in a meander maze figure 1.2, as illustrated right in figure 5.4. Advanced Sudoku however introduces fewer clues, requiring testing contingencies, and hence has a simply-connected tree maze as in example 5.1.

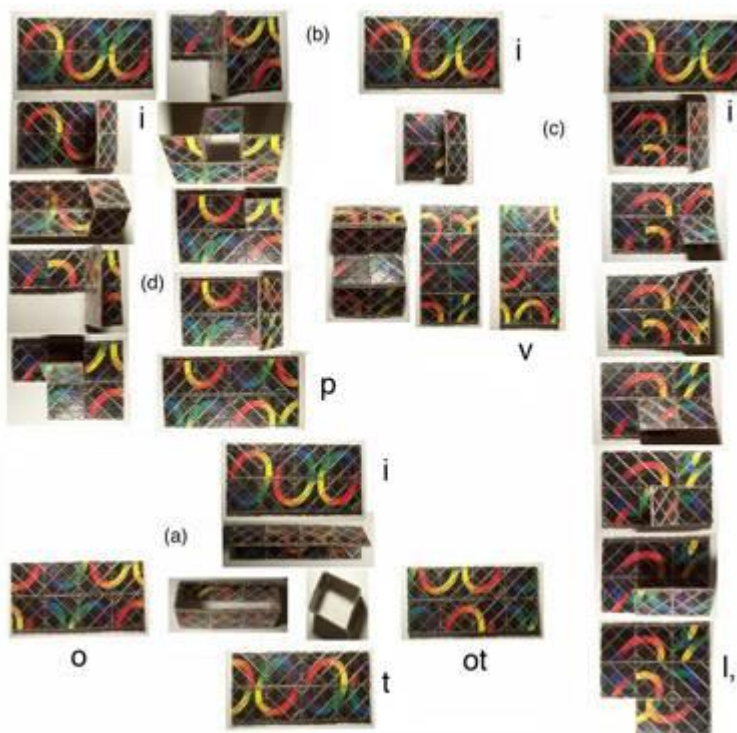


Figure 5.5a: Four sequences of geometric moves in the Rubik 3-rings puzzle.

Example 5.5 The Rubik Three Rings Puzzle

The Rubik 3-rings puzzle consists of a set of eight diagonally grooved plates held together by nylon strings woven over three successive plates in a circuit in overlapping succession, so that the plates can be folded along certain axes joining the plates, changing the way the strings link the plates and creating new puzzle geometries. The aim of the puzzle is to fold the plates into a new arrangement where the three unlinked rings have seemingly impossibly become linked. This is possible because the reverse sides of the plates have pieces of a second image of the three rings linked through one another as shown in the heart shape in the centre of figure 5.5c, associated with an L-shaped geometry differing from the rectangular starting position.

The puzzle presents an intriguing mix of geometrical and topological constraints, the weaving of the strings fixing the geometry of the hinged shapes, within the basic topology of a ring of plates

in which some, but never all, of the plates are able to be hinged out of the loop, at least temporarily.

The weaving of the strings itself presents an interesting topological puzzle, which enables the eight rings in the rectangular configuration to be transformed in every possible way that retains their overall ring structure. There are 8 clockwise permutations of the plates, two directions of orientation around the ring and four orientations the square plates can adopt collectively. This gives $8 \times 2 \times 4 = 64$ possible states of the rectangle, however we need to divide this figure by 2, since moving four steps round the ring rotates the whole rectangle through 180° , if the top row is coded *abcd* and the bottom row is an inverted *efgh*. The way the strings are woven enables all of the 32 possible states to be reached, although this might seem impossible from the way they are woven.

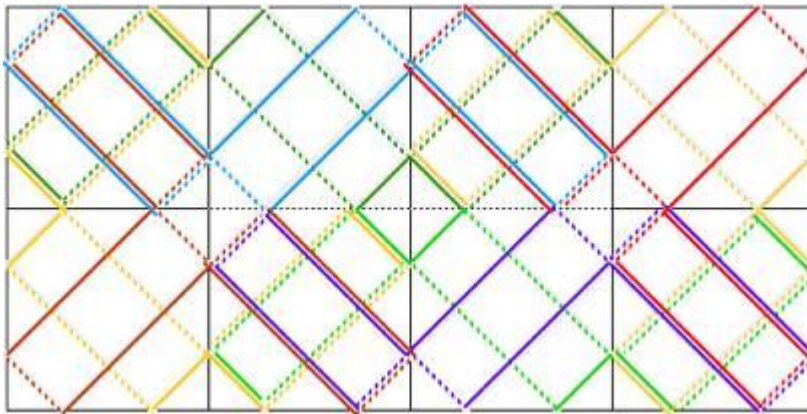


Figure 5.5b Colour-coded weaving of the 8 closed loop string pairs holding the 8 3-ring puzzle plates in a loop.

In figure 5.5b is shown the weaving of the 8 looped string pairs in a 4x2 arrangement, which remain unknotted throughout, although alternate plates have a double winding, with strings linking to 2

adjacent plates, spanning 3 in all. There are no vertical connections in the centre four plates, so the 8 plates form a ring.

To solve the puzzle requires negotiating a series of geometrical transformations, some of which lead to cul-de-sacs, however there are four sequences of transformations illustrated in figure 5.5a, which lead to a rearrangement of the rectangular arrangement. In (a) the ring is folded and becomes a literal ring of 8 plates, which can be unfolded to form four rectangular states involving 3 transformations from the identity. In (b) the plates can be folded together above and then unfolded in the vertical direction from below, effectively rotating the plates through 90° . In (c) a sequence of moves takes the rectangle to a scrambled form of the L or heart-shape of the final solution, which can then be refolded from the other side of the L to gain a different transformation of the rectangle. There is a mirror image of this entire sequence, which forms an inverse transformation. Finally in (d) there is another move, which results in a new set of configurations, resulting in 7 transformations in all.

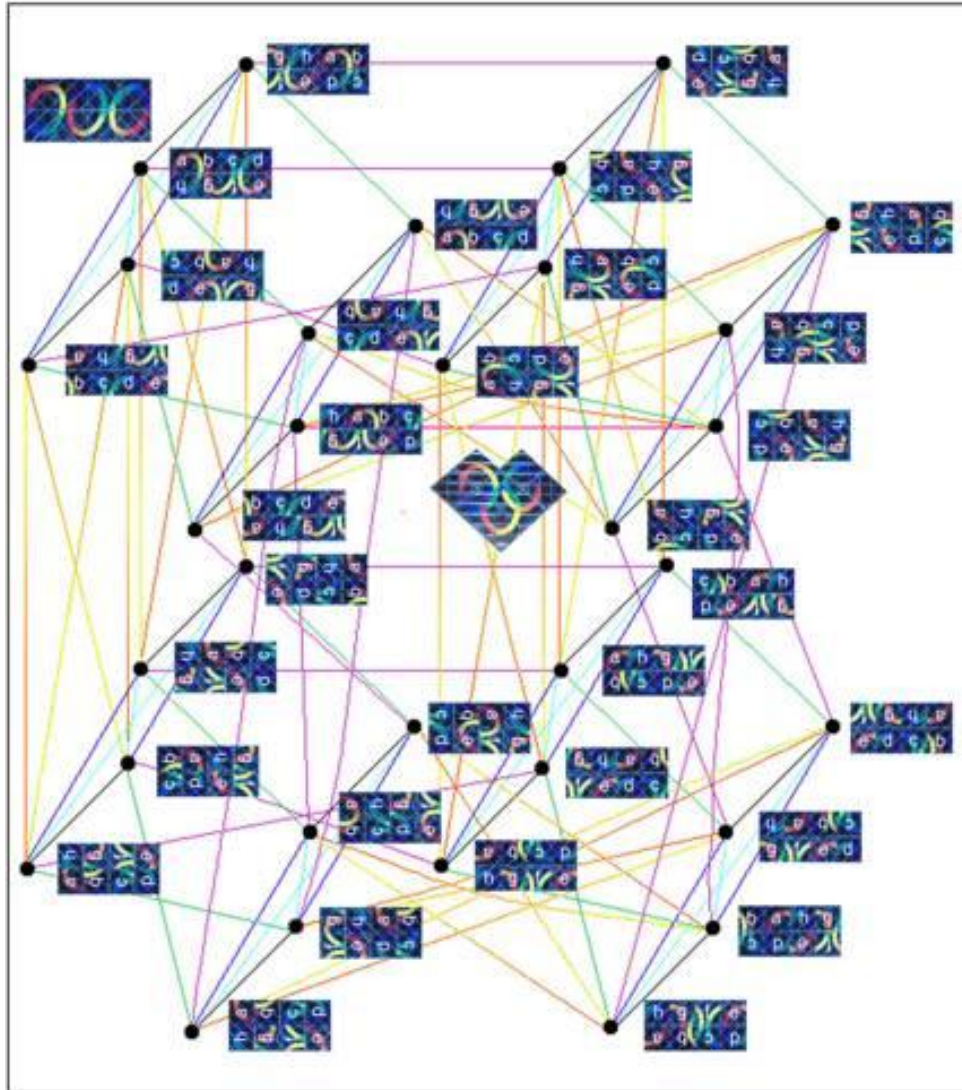


Figure 5.5c: The state space graph of the rectangular configurations presented as a non-commutative graph assembled on the 4-D hypercube of 32 states.

Because the geometry of these moves is complex, we have a non-trivial puzzle which has a state space graph which has only 32 nodes corresponding to the 32 possible transformations of the 8 plates above, so we see another example of the trade off between individual transition complexity and state graph size.

To analyze the state space graph, the seven possible transformations of the rectangular configuration, a Matlab simulation was made of each of the geometrical transformations and this was used to check the definition of each of the 7 transformations arising from each node. The result is shown in figure 5.5c.

Although this is a richly interconnected graph with a large number of loops, navigating from one position to another is still difficult because several of the operations fail to commute in diverse ways, causing operations performed out of order to arrive at unfamiliar destinations.

The seven transformations are colour-coded and the state of each node is illustrated and coded using the abcdefgh notation above. Each of the pairs of edges forming a parallelogram in the graph commute while the others do not. Each of the transformations are self-inverses, except for red-yellow shaded ones passing through the heart-shape intermediate, whose two forms are mutual inverses. There is a corresponding 32 node state graph for the heart-shapes, each of which is connected to two rectangles through inverse transformations, two of which emerge from each rectangle.

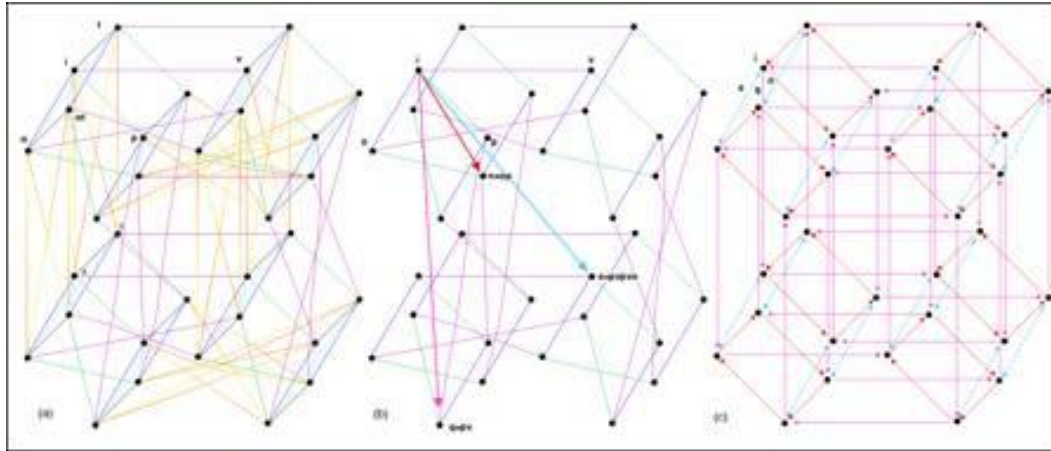


Figure 5.5d Stages in disentangling the transformation group. (a) Graph of the 7 geometric operations t, o, p, v, r, l and op . (b) Reduced graph with 3 generators o, p, v and defining new generators n, q and e . (c) Rearrangement of vertices using the new generators results in a non-commutative hypercubic Cayley graph.

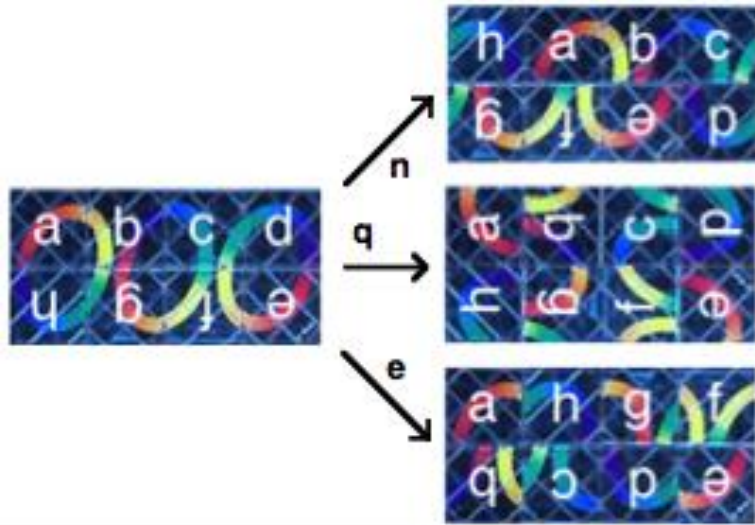
To decode the actual 32-member group^[25], first we eliminate redundant operations. Tracing the connections in figure 5.5d(a) we can see immediately that three of the key geometrical operations including the simplest (t) and the ones that pass through the heart solution (l and r) are composites of the others: $t=opop, l=povp, r=opvp$.

Removing these yields the graph in (b) and a group with a presentation^[26]:

$$G = \{o, p, v : o^2 = p^2 = v^2 = i, (op)^4 = (ov)^4 = (vp)^4 = i, opop = ovov, ov = opvp\} \quad [5.5.1]$$

In so doing, we have eliminated the very geometrical transformations that enabled us to get to the heart shaped solution. We should note that a similar description could be mounted of all the transformations of the 32 hearts.

Examining the symmetries of the plates in figure 5.5c however, we can easily see that more natural operations are available which are composites of o, p , and v but represent fundamental symmetries of the rectangle.



We define three new transformations of the rectangle (RH composition):

1. $n=op$ Moves all the plates cyclically right by one step, rotating plates 180° when they move around the end of the ring.

Four such moves rotate the rectangle through 180° leaving the puzzle unchanged.

2. $q=pv$ Rotates alternate plates 90° clockwise and anti-clockwise.

3. $e=vpvo$ Reverses the orientation of the ring of 8 plates leaving the top left and bottom right plates unchanged.

Given these generators, we can present the group G , with center ^[27] $\{i, nn = t, qq, nnqq\} \cong C_2 \times C_2$ as:

$$G = \{n, e, q : n^4 = q^4 = e^2 = i, qe = eq, qn = nq^{-1}, ne = en^{-1}\} \quad [5.5.2]$$

We can then rearrange the vertices to reflect the symmetries and arrive at a non-commutative, hypercubic Cayley graph ^[28] for the transformation group as in figure 5.5d(c). If we use the notation $A \times_{-1} B$ for a semi-direct product ^[29] action by inverting elements: $ba = ab^{-1}$, then, from the relations, G can be characterized by: $G \cong D_4 \times_{-1} C_4 \cong (C_2 \times_{-1} C_4) \times_{-1} C_4$ since $eq = qe = qe^{-1}$ and $qn = nq^{-1}$, where D_4 is the dihedral group of transformations of the square and C_n is the cyclic group of order n (e.g. of integers modulo n under addition).

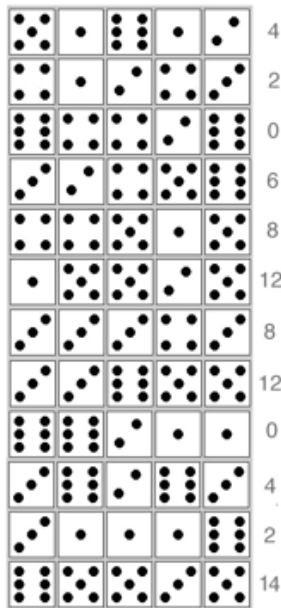


Figure 5.6 A sample of die throws from “Petals Around the Rose”
 Example 5.6 Petals Around the Rose

“Petals Around the Rose” ^[30] is a puzzle that is famous for its account of Paul Allen and Bill Gates introduction to it in a crowd returning from a computing conference in 1977, in which Bill was the last active player in the group to discover the rule. The game has only two clues. One is that the answers are all even, which becomes obvious after a few throws, and the other is “Petals around the Rose”, which is significant. No one is supposed to reveal anything more than the answer to a throw – never the rule itself.

The problem to be solved, rather than one of deductive thinking as in the zebra puzzle is one of lateral thinking, faced with a seemingly irregular rule. The state space of the puzzle now consists of all the lateral shifts of thinking the subject might imagine, so it cannot be defined precisely in the way the previous examples were. There are a great variety of rules which could be applied, some involving adding or multiplying the values

on the faces, others counting how many die of a given value appear, others dealing with the geometry of the faces, the way the dice fall or the order of them in sequence, but each of these conjectures form part of the topography of the state space which the subject explores till they see a contradiction, until eventually they discover the rule, which for convenience I will print upside down in light grey below, so you can read it only if you can't deduce it from the instances in figure 5.6.

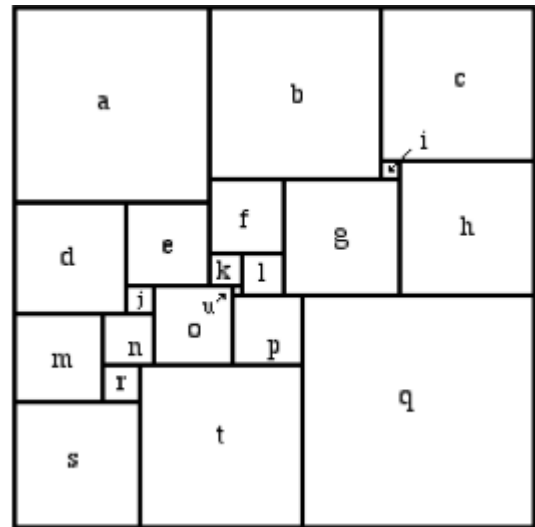
Critical to the irregularity is that the rule uses only partial information from the dice. This information is highlighted both by the high scores and the very low scores, which are over-represented in the list in the figure in the interests of quick analysis.



Figure 5.7a The unique simple perfect square of order 21 (the lowest possible order).

Example 5.7 Squaring the square and Magic Squares

Not all puzzles involve a state space. Some are better solved in one step, or a single defined process, e.g. by defining a system of equations. One such example is squaring the square^{[31] [32]}, where we are asked to find the relative dimensions of the tiled unequal squares fitting into a single large square in figure 5.7.



This is an ideal candidate for using symbolic manipulation to take the boredom out of the algebra. We first investigate the geometry and compare a series of vertical and horizontal side lengths until we have generated enough equations for a unique solution, and set the smallest square to a suitable base number.

The Matlab symbolic toolbox provides an ideal solution platform:

```
syms a b c d e f g h i j k l m n o p q r s t u
S=solve('l=k+u,f=k+l,g=f+l,h=g+i,c=h+i,b+g=c+h,o=p+u,o=j+n,a=d+e,b+c=f+g+h,s=m+r,t=r+s,
q=p+t,a+b+c=d+e+f+g+h,a+b+c=s+t+q,a+b+c=m+n+o+p+q,a+d+m+s=a+e+j+n+t,a+d+m+s=c+
h+q,c+h+q=a+e+o+t,a+e+o=b+f+l+p,b+i=f+g,e+k=j+o+u,o+e=d+n,d+j=m+n');
C=struct2cell(S);
u=2;
for i=1:21
    fprintf('%c=%2.0f ',char(96+i),eval(C{i}));
end
a=42 b=37 c=33 d=24 e=18 f=16 g=25 h=29 i=4 j=6 k=7 l=9
m=19 n=11 o=17 p=15 q=50 r= 8 s=27 t=35 u=2
```

However such puzzles are neither common, nor as popular as those which require a conceptual hunt through a space of possibilities, and in this case the problem is unique, being the only simple perfect square of order 21 (the lowest possible order), discovered in 1978 by A. J. W. Duijvestijn^[33].

8	1	6
3	5	7
4	9	2

Figure 5.7b Lo Shu the unique 3x3 magic square is associative and generated by the Siamese method..

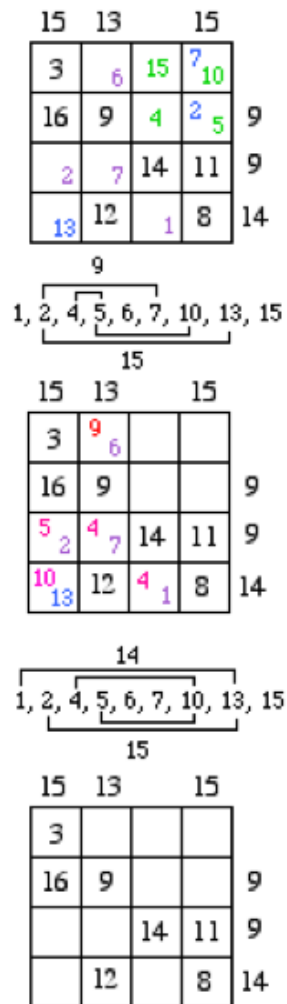
To explore the problem of puzzle generation in numeric puzzles we can explore the problem of magic squares^[34]. A magic square is a square array of numbers, consisting of the distinct positive integers 1, 2, ..., arranged such that the sum of the numbers in any horizontal, vertical, or *main* diagonal line is always the same number, known as the magic constant $\frac{n(n^2+1)}{2}$. The unique 3x3 square was known to the ancient Chinese as Lo Shu. This is also associative if pairs of numbers symmetrically opposite the centre sum to n^2+1 . If *all* diagonals (including those obtained by wrapping around) of a magic square sum to the magic constant, the square is said to be a panmagic square also called a diabolic square.

It is an unsolved problem to determine the number of magic squares of an arbitrary order, but the number of distinct magic squares (excluding those obtained by rotation and reflection) of order 1-5 are 1, 0, 1, 880, 275305224, and an estimate of order 6 is 1.77×10^{19} using Monte Carlo simulation and methods from statistical mechanics. The number of distinct diabolic squares of order 1-5 are 1, 0, 0, 48, 3600.

Given the unbounded number of solutions one would expect there exists simple regular algorithms for generating magic squares and this is the case. The Siamese method consists of placing a 1 anywhere and placing 2, 3 etc. successively up the right hand diagonal (vector (1,1)) moving one down (break vector (0,-1) if we hit a filled square. Lo Shu in figure 5.7b can be seen to be generated in this way. The Siamese method will also generate diabolic magic squares of order $6k \pm 1$ with vector (2,-1) and break vector (1,1).

Figure 5.7c A sample 4x4 square puzzle made by removing magic square entries, has a simple tree maze with two branch points, corresponding to contingencies in the bottom left and top right corners.

Magic squares can be used to generate Sudoku-like puzzles with state space tree mazes of varying complexity. In figure 5.7c is shown a sample 4x4 diabolic magic square in which over half the entries have been omitted. The entries outside the square give the remainder when the existing entries are subtracted from the magic constant of 34. In the first stage the bottom-left entry is used to compare information from its row and column. This implies a corresponding set of contingencies in linked rows and columns leading to an impasse for one (the 9 in position



(1,2). This information can now be used to perform the same analysis for the top-right entry leading to the solution. Once again the numeric puzzle leads to a path-connected graph, in this case a tree with two branch points, giving the puzzle an underlying topological basis.

Because the number of possible magic squares grows so rapidly, increasing the size of the square and reducing the number of given entries can rapidly lead to too many contingencies to make an interesting and 'doable' puzzle because of the load of multiplying contingences and the repetitious simple arithmetic involved.

Example 5.8 2-D and 3-D Tiling with Polyminoos



While some puzzles have one solution, which might be solved, like squaring the square, by a system of equations, an abstract proof, or a single algorithm, others have many possible solutions, often with their own internal irregularities, which require a brute force computational approach to find all the variations. One of the most persistent and intriguing types of puzzle to many people are geometrical tiling puzzles constructed out of systematic geometric variants, such as pentaminoes (all 12 configurations of 5 attached cubes in 2-D), the pieces of the

soma cube (all 7 non-linear pieces of composed of 3 or 4 cubes in 3-D) and the Kwazy quilt made of all combinations of circles stellated with up to six regular apices.

Figure 5.8a Anti-clockwise from top: 6 variants of the soma cube, viewed front and back, 6 variants of the 'Lonpos Pyramid', one of only 2 possible 3x20 pentamino solutions, 'Kwazy Quilt', and compound happy cube and hypercube illustrate tiling puzzles with multiple solutions.

The soma cube was invented by Piet Hein^[35] the scientist, artist, poet and inventor of games such as hex, during a lecture on quantum mechanics by Werner Heisenberg. There are 240 essentially distinct ways of doing so, as reputedly first enumerated one rainy afternoon in 1961 by John Conway and Mike Guy.

However, if we count the internal symmetries of individual pieces within themselves, i.e. $3 \times 2^3 = 96$ and the $6 \times 4 \times 2 = 48$ symmetries of the whole cube we arrive at $96 \times 48 \times 240 = 1105920$. This might be compared with the maximum number of distinct, possibly non-tiling arrangements of the pieces in space $7! \times 24^7 / 96 = 2.4 \times 10^{11}$. Because a subject assembles a cube using less tractable pieces first, it is relatively easy to find a solution, and to navigate in the maze of solution space

using geometrical intuition using as many back step as necessary to retreat from cul-de-sacs near completion. A variety of other geometrical shapes can also be made with the soma pieces, having varying degrees of constraint and hence difficulty.

Likewise the 'Lonpos Pyramid' uses a subset of spherically-based 2-D polyminoes of sizes 3, 4 and 5 to build a pyramid, as well as rectangular solutions. Although the pieces are planar, the pyramidal solutions involve interlocking pieces aligned horizontally, vertically and obliquely. Since most are horizontal it is generally easier to solve from the apex of the pyramid, which places strong local constraints on the pieces to be used.

The 12 2-D pentaminoes, known from the 19th century, are capable of tiling several rectangles of area 60 units, as well as other shapes, such as using 9 to tile versions of the individual pieces expanded 3 times in size (45 units area). The number of rectangular solutions are: 6×10 (2339), 5×12 (1010), 4×15 (368), 3×20 (2). This might be compared with something like 3.2×10^{16} independent orderings and orientations of the 12 pieces. The rectangular puzzles each have similar difficulty, despite the varying number of solutions, because the narrower rectangles place more constraints on the feasible partial tilings.

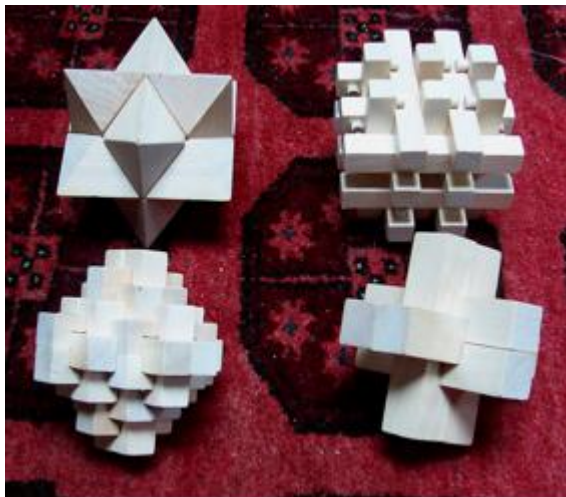


Figure 5.8b Four wooden interlocking puzzles.

The popularity of such puzzles with both adults and children, including their variants in wood puzzles (left) that generally have only one way of being assembled, illustrates a strong theme involving the geometry of mental rotation, the topology of navigating a path in abstract solution space, and a preference for dealing with mathematical problems which have a strong sensory basis, are capable of direct manipulation and promote lateral thinking, to open unperceived avenues and avoid tunnel vision, as well as deductive reasoning. These themes all support a

linkage between puzzles and gatherer hunter skills, which have evolved over long epochs and stand diametrically opposed to the dominance abstract linguistic-based axiomatic manipulations have in proofs in classical theoretical mathematics. The gulf between these perspectives becomes ever more acute in an era when pocket calculators and laptop computers are making redundant many of the arithmetic skills of mental calculation we have come to assume go hand in hand with civilization.

Example 5.9 Peg Solitaire as a large State Space with Internal Symmetries

Table 5.9 Successive board positions in peg solitaire ^{[36][37]}

Peg solitaire has a long and colourful history, being spuriously attributed both to native Americans and to a French aristocrat imprisoned in the Bastille, but can be specifically traced back to the court of Louis XIV in 1697, from when its repeated representation in art shows it had wide popularity. In the classical game, the board is filled with pegs except for the central position, and the aim is by jumping over and removing successive pieces, to end with a single peg remaining in the centre. The English board forms a cross comprising 33 holes, as shown in figure 5.9 and admits multiple solutions, but the European version with four extra pegs does not admit a classical solution, so we shall consider the English game, although there are also many puzzle variants.

oles	Moves	Positions	Winning	Terminal	Tot Positions	Dead Ends
1	0	1	1		1	0
2	1	1	1	0	4	0
3	2	2	2	0	12	0
4	3	8	8	0	60	0
5	4	39	38	0	296	0
6	5	171	164	0	1,338	0
7	6	719	635	1	5,648	32
8	7	2,757	2,089	0	21,842	0
9	8	9,751	6,174	0	77,559	0
10	9	31,312	16,020	0	249,690	0
11	10	89,927	35,749	1	717,788	280
12	11	229,614	68,326	1	1,834,379	31,920
13	12	517,854	112,788	0	4,138,302	0
14	13	1,022,224	162,319	5	8,171,208	386,416
15	14	1,753,737	204,992	10	14,020,166	1.82E+07
16	15	2,598,215	230,230	7	20,773,236	5.24E+07
17	16	3,312,423	230,230	27	26,482,824	5.69E+08
18	17	3,626,632	204,992	47	28,994,876	3.64E+10
19	18	3,413,313	162,319	121	27,286,330	3.80E+11
20	19	2,765,623	112,788	373	22,106,348	8.52E+12
21	20	1,930,324	68,326	925	15,425,572	1.96E+14
22	21	1,160,977	35,749	1,972	9,274,496	3.72E+15
23	22	600,372	16,020	3,346	4,792,664	5.31E+16
24	23	265,865	6,174	4,356	2,120,101	6.05E+17
25	24	100,565	2,089	4,256	800,152	4.41E+18
26	25	32,250	635	3,054	255,544	2.16E+19
27	26	8,688	164	1,715	68,236	8.25E+19
28	27	1,917	38	665	14,727	1.36E+20
29	28	348	8	182	2,529	2.11E+20
30	29	50	2	39	334	1.05E+20
31	30	7	1	6	32	1.63E+19
32	31	2	1	2	5	8.17E+16
		23475688	1679072	21111	187636299	5.77117E+20

A brute force attack on the possible number of positions in n moves gives the sequence in table 5.9. The total number of reachable board positions is the sum 23,475,688, while the total number of possible board positions is $2^{33} / 8 = 10^9$ when symmetry is taken into account. So only about 2.2% of all possible board positions can be reached starting with the center vacant. 'Tot Positions' ignores the symmetries of board rotations and reflections which are factored out in 'Positions'. Counting successive board positions into a cumulative set of plays, there are 577,116,156,815,309,849,672 or 5.7×10^{20} different complete game sequences, of which 40,861,647,040,079,968 or 4×10^{16} are solutions. Thus although there are theoretically a huge

number of solutions, the probability of finding one at random is about 1 in 10,000. Until a player finds a winning strategy, they tend to initially move in a haphazard way, hoping to arrive fortuitously at an end-game they can resolve more easily, and are thus unlikely to find a solution.

Since any jump exchanges 2 pegs and a hole with 2 holes and a peg and the start position exchanges holes and pegs as well, there is a symmetry between start and finish, which means that exchanging pegs and holes and playing backwards from the finish will provide a complementary strategy to the original. This can be seen from the symmetry in the winning positions in table 5.9. One appealing winning sequence first collapses the cross to one move off a smaller central diamond game before closing in with a grand circuit. The complement to this game counter-intuitively removes the centre diamond before the arms of the cross arriving back at the centre.

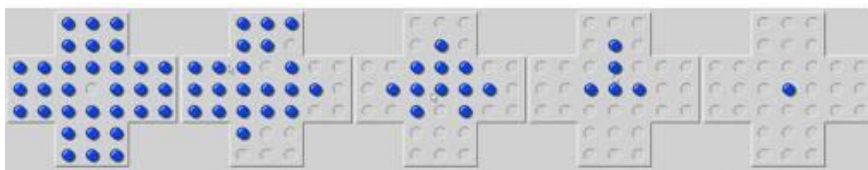


Figure 5.9 Five stages of a winning game of peg solitaire which first reduces the game to one move off a

smaller diamond-shaped version of the game before making a grand tour leaving a single T which collapses to the solution. The reverse of this game with pegs exchanged for holes gives a second counter-intuitive solution in which the central diamond is first removed leaving the peripheral parts of the cross, finishing with a move to the centre. Other games win by an amorphous strategy.

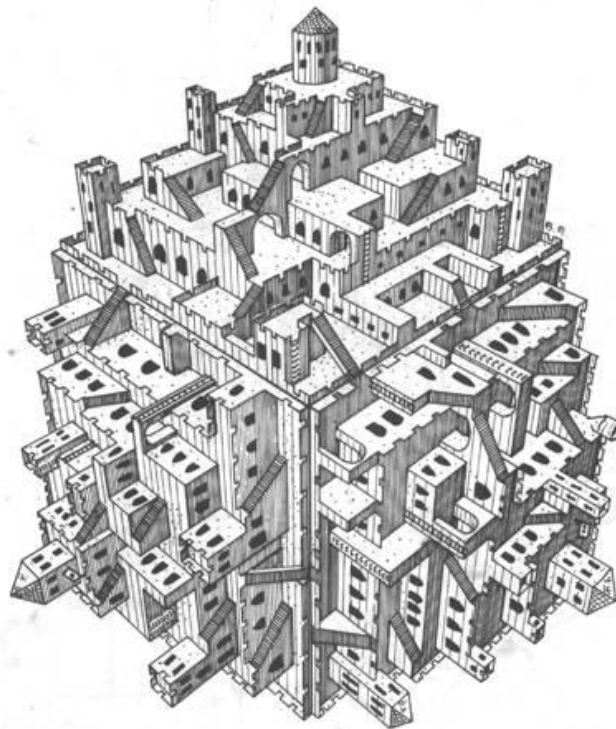


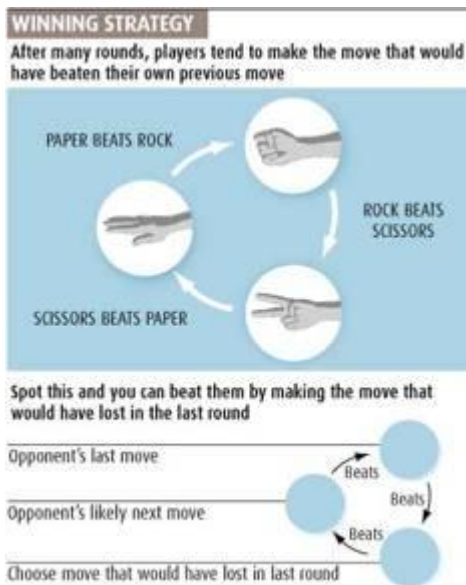
Figure 5.10 Cover maze from Supermazes^[38]

Example 5.10 Mazes as Topological Puzzles

Finally we return to mazes, which, in addition to underlying the solution space of every puzzle, constitute the most ancient and intrinsically topological form of puzzle known. The state space of the maze is precisely the set of positions negotiated in traversing it. Although, as in the example of figure 5.8 they may have a complex topology of overpasses and underpasses in the manner of knot theory, from the subjects point of view this is secondary to the path connecting the start and finish, so the structure of a maze is determined by its path-connected graph, which is trivially a line for a meander maze (figure 1.2), a tree for a simply-connected maze, which can then be traversed however laboriously by a systematic right hand rule following all cul-de-sacs to exhaustion, however in a maze with loops although there may be more than one path, the strategy needs to avoid becoming locked in cycles.

While we are told Theseus had to follow Ariadne's thread to return from slaying the Minotaur, this may have been merely to avoid becoming disoriented in the dark winding passage of the labyrinth, because the image of coins from Knossos from figure 1.2 suggests this, like Roman and floor mazes in many cathedrals was a simple meander maze requiring no choices, but just a long tortuous walk, in stark contrast to the duplicitous topological paths in the wilderness humanity has negotiated, since the dawn of history and the equally elaborate paths in state space we have discovered in analyzing the above puzzles.

Some of these state spaces like the Rubik revenge with 7.4×10^{45} and even Solitaire with 5.7×10^{30} are huge, but Go with 4.63×10^{170} board positions and Chess with an estimated $10^{10^{50}}$ possible games^[39] and between 10^{40} and 10^{120} board positions at the 40th move surely take the breath away and make one realize the Machiavellian theory of the evolution of intelligence, based on social strategic bluffing for sexual favours and personal power in a complex human society of many players has an invincible and convincing ring to it!



Example 5.11 Scissors-Paper-Stone Topological bifurcation as a basis for a complementary strategy space. Scissors-paper-stone is a game consisting of an apparently irresolvable cyclic transitive relationship of dominance. There is thus no specific winning set of moves and winning play depends on a bifurcation between two complementary strategies of defense and attack. The defensive strategy is to randomize your moves as completely as possible so the opponent has no pattern they can fix on to take advantage. The complementary attacking strategy is to deduce the opponent's pattern and choose the move that will capture the move anticipated by the pattern. Various statistical deviations in human behavior can also be capitalized on. The choices are commonly skewed rock gaining 36% paper 30% and scissors 34% so a player can take advantage of the skew. Players also tend to pick moves that would have beaten their previous move, so

choosing a move which your opponent would have just defeated is a paradoxically winning strategy^[40].

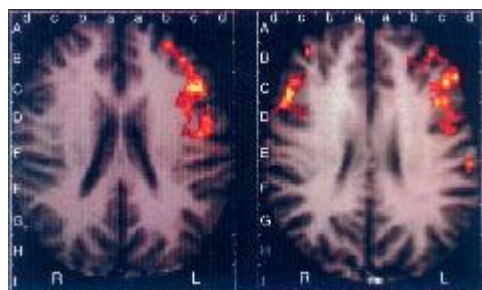
6: The Brain's Eye View of Mathematics

Despite the strides of such techniques as the electro-encephalogram and functional magnetic resonance imaging, research into how mathematics is processed in the brain is still in its infancy. Evidence from cultural and development studies and the effects of brain injury, are rapidly being complemented by research to elucidate the localization in the brain of various aspects of mathematical reasoning, however these have so far dealt mainly with basic level mathematical skills such as raw numeracy – e.g. comparing numbers and tasks such as mental rotation, which are already the fare of psychological experiment.

Figure 6.1 Sex differences in mathematical performance tests are not paralleled in verbal performance tests^[41].

Views of the basis of mathematical reasoning in the brain vary widely. At one extreme is the notion that numeracy is a hard-wired genetically based trait^[42] located in the left parietal lobe (related to finger counting) and even more basic than language.

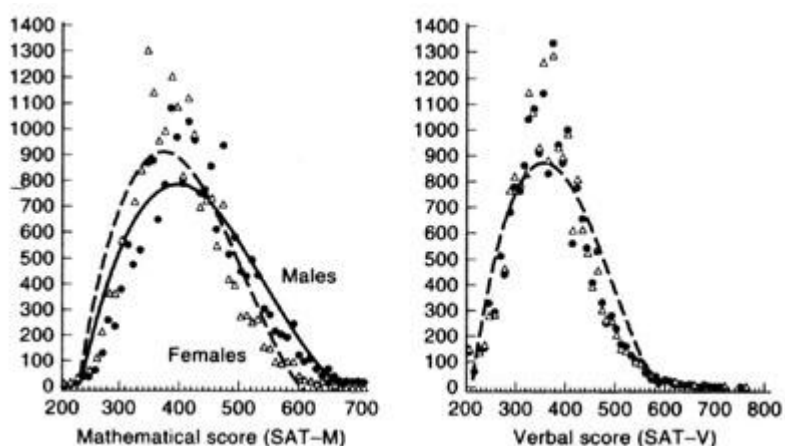
On a somewhat different tack, Stanislas Dehaene^[43] the founder of the triple-code model discussed below, sees both hemispheres being involved in manipulating Arabic numerals and numerical quantities, but only the left hemisphere having access to the linguistic representation of numerals and to a verbal memory of arithmetic tables.



There is some evidence for a genetic basis in mathematical ability, in subtle gender differences in performance at mathematical tasks^[44], which is not reflected in language acquisition (figure 6.1) despite the significantly different degree of language lateralization in male and female brains (figure 6.2).

Figure 6.2 Sexual differences in language processing^[45].

These mathematics skill differences appear to be real and not just based on differences of educational opportunity. The most comprehensive study published in Science in 1995 found that in maths and science in the top ten percent, boys outnumbered girls three to one. In the top one percent there were seven boys to each girl. By contrast in language skills there were twice as



many boys at the bottom and twice as many girls at the top. In writing skills girls were so much better, boys were considered ‘at a rather profound disadvantage’^[46].

Contrasting a biologically-based view of numeracy are studies which demonstrate cultural differences in the way the same numeracy problem is presented, such as those comparing Chinese and English speakers (figure 6.3). Whereas in both groups the inferior parietal cortex was activated by a task for numerical quantity comparison, such as a simple addition task, English speakers, largely employ a language process that relies on the left perisylvian cortices for mental calculation, while native Chinese speakers, instead, engage a visuo-premotor association network for the same task. Also raising doubts about the genetic basis of numeracy is the discovery of the Amazonian Pirahã^[47] who live without any notions of numbers more specific than ‘some’ and cannot count. This is consistent with the fact that apart from some savant’s and geniuses such as Ramanujan^[48], most people have a digit span of only seven, and a mental calculation capacity vastly inferior to a simple pocket calculator.

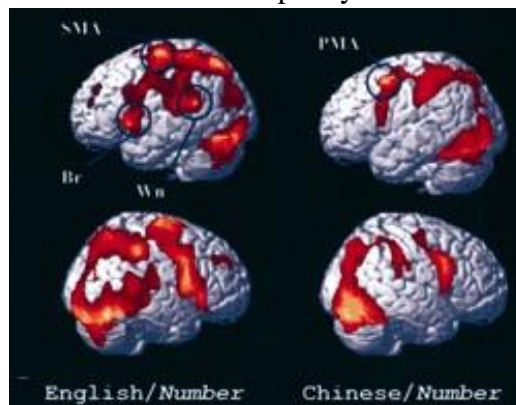


Figure 6.3 Language-based differences in mathematical processing^[49].

While some people from Noam Chomsky’s generative grammar^[50] to Stephen Pinker’s “Language Instinct”^[51] contend that language is a genetically based evolutionary trait, other models of language^[52] see the genetic basis as more generalized and that spoken languages have ‘taken-over’, as increasingly efficient systems more in the manner of a computer virus through their cultural evolution by colloquial use. This view has support in the much more rapid evolution of languages and the fact that, while we do not know how long ago people first began speaking, written language has only a short human history, consistent with our reading skills being an adaption of more generalized visual pattern recognition systems. Since numeracy and mathematics depend prominently on Arabic numerals, although having a basis in analog comparison and finger counting, the visual symbolic basis of mathematics is likewise likely to be a cultural adaption.

The brain consists of two hemispheres connected by a bunch of white matter called the Corpus callosum. Ever since split-brain experiments on monkeys there has been a fascination with the idea that the two hemispheres in humans may have different or complementary functions, stemming partly from the knowledge coming originally from war injuries and strokes that injury to the ‘dominant’ left hemisphere which is usually contra-laterally connected to the use of the right hand, is selectively devoted to language typified in Broca’s area of the frontal cortex which facilitates fluent speech and Wernike’s area of the temporal cortex, which mediates meaningful semantic constructions. Although this result came predominantly from men and brain scans on both sexes have subsequently showed that language acquisition in women is more bilateral than in men, the idea that the two hemispheres had complementary functions captured the imagination of neuroscientists.

There is some evidence generally for this idea with music and creative language use having a partially complementary modularization to structured language. This in turn led to the idea that the more structured aspects of mathematics, such as algebra, and the more amorphous entangled aspects such as topology might be processed in different ways in complementary hemispheres. While this idea is appealing, there are few actual experiments that have tested the idea, and brain scan studies have tended to concentrate on elementary mathematical skills, which psychologists and neuroscientists can test on a wide variety of subjects researching basic brain skills, such as mental arithmetic and mental rotation, rather than complex abstract procedures.

Theories about how mathematical reasoning is processed gravitate on common sense ideas linking specific sensory modalities, known linguistic capabilities and general principles of frontal cognitive processing to generate parallel processing models of brain-related modalities having a natural affinity with mathematical reasoning.

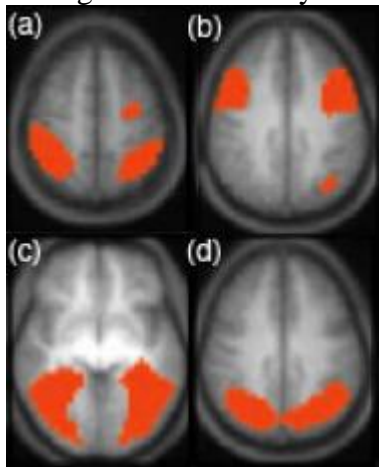
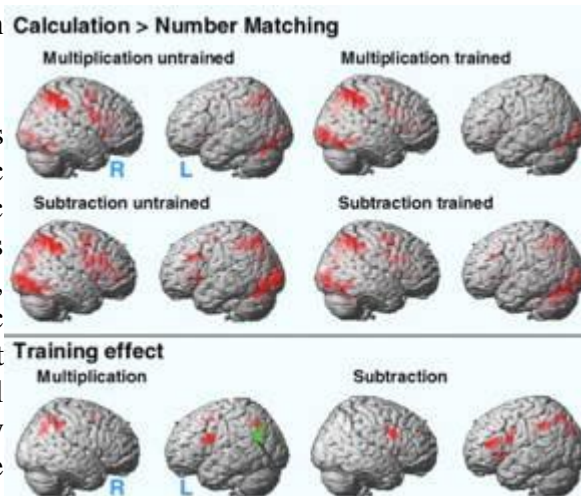


Figure 6.4 Triple-code model^[53] of numerical process in has support from independent component analysis of fMRI scans of mental addition and subtraction revealing four components. (a) bilateral inferior parietal component may reflect abstract representations of numerical quantity (analog code) (b) left perisylvian network including Broca's and Wernicke's areas and basal ganglia reflecting language functions. (c) ventral occipitotemporal regions belonging to the ventral visual pathway and (d) secondary visual areas consistent with a visual Arabic code.

For example the triple-code model^[54] (figure 6.4) of numerical processing proposes that numbers are represented in three codes that serve different functions, have distinct functional neuro-architectures, and are related to performance on specific tasks. The analog magnitude code represents numerical quantities on a mental number line, includes semantic knowledge regarding proximity (e.g., 5 is close to 6) and relative size (e.g., 5 is smaller than 6), is used in magnitude comparison and approximation tasks, among others, and is predicted to engage the bilateral inferior parietal regions. The auditory verbal code (or word frame) manipulates sequences of number words, is used for retrieving well-learned, rote, arithmetic facts such as addition and multiplication tables, and is predicted to engage general-purpose language modules, associated with memory and sequence execution. The visual Arabic code (or number form) represents and spatially manipulates numbers in Arabic format, is used for multi-digit calculation and parity judgments, and is predicted to engage bilateral inferior ventral occipito-temporal regions belonging to the ventral visual pathway, with the left used for visual identification of words and digits, and the right used only for simple Arabic numbers.

In research focusing on the intra-parietal regions contrasting number comparison with other spatial tasks^[55], number-specific activation was revealed in left IPS and right temporal regions, whereas when numbers were presented with other spatial stimuli the activation was bilateral^[56].

Figure 6.5 Unpracticed and learned tasks in multiplication and subtraction are contrasted^[57].



Further support for the triple-code model comes from studies of learning complex arithmetic (multiplication)^[58], where left hemispheric activations were dominant in the two contrasts between untrained and trained condition, suggesting that learning processes in arithmetic are predominantly supported by the left hemisphere. Activity in the left inferior frontal gyrus may accompany higher working memory demands in the untrained as compared to the trained condition. Contrasting trained versus untrained condition a significant focus of activation was found in the left angular gyrus. Following the triple-code model, the shift of activation within the parietal lobe from the intraparietal sulcus to the left angular gyrus suggests a modification from quantity-based processing to more automatic retrieval. A second study involving learning multiplication and subtraction supports similar conclusions (figure 6.5). This trend suggests that learned mathematical tasks of this kind become committed to linguistic memorization, once they are mastered.

In contrast with this, an experiment where subjects were asked to analyze a simple mathematical relationship^[59], e.g. $x = A$, $B = A + 6$, $C = A + 8$ by either forming a number line picture, or constructing the left side of a solving equation e.g. $x + (x + 6) + (x + 8) = 5$, in both cases visual processing areas were activated and there were no significant differences in processing in language areas. This suggests visual processing areas are involved in forming equations, at least unfamiliar newly presented ones.

An intriguing study, which has more implications for advanced mathematics, where real conjectures are examined and proved, or found false, examined brain areas activated when true and false equations were presented to the subject^[60]. This study found greater activation to incorrect, compared to correct equations, in the left dorsolateral prefrontal left ventrolateral prefrontal cortex, overlapping with brain areas known to be involved in working memory and interference processing.

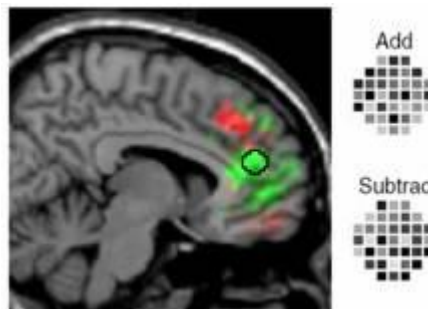


Figure 6.6 Prefrontal areas activated differentially when incorrect mathematical equations are presented.

Extending this into the geometrical area and specifically with gifted adolescents is a study of mental rotation (figure 6.8) involving images such as 3-D polyminoes. In contrast to many neuroimaging studies, which have demonstrated mental rotation to be mediated primarily by the right parietal lobes, when performing 3-dimensional mental rotations,

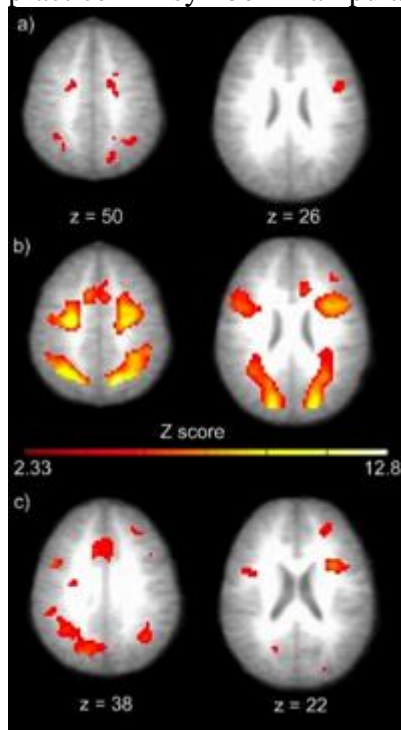
mathematically gifted male adolescents engage a qualitatively different brain network than those of average math ability, one that involves bilateral activation of the parietal lobes and frontal cortex, along with heightened activation of the anterior cingulate.

Figure 6.7 Differential activation of the medial prefrontal cortex can predict a person's intention to add or subtract two numbers^[61].



It has also become possible to teach a computer to distinguish subjects' intention to add or subtract two numbers, using analysis of detailed differential activation of the medial prefrontal cortex, giving predictions which are 70% accurate (figure 6.7).

A brain imaging study of children learning algebra (simple linear equations)^[62], shows that the same regions are active in children solving equations as are active in experienced adults solving equations, however practice has a more striking adaptive response in children. As with adults, practice in symbol manipulation produces a reduced activation in prefrontal cortex area.



However, unlike adults, practice seems also to produce a decrease in a parietal area that is holding an image of the equation. This finding suggests that adolescents' brain responses are more plastic and change more with practice.

Figure 6.8 Mental rotation: Above average subjects, middle gifted subjects, below the difference in activation between the groups^[63].

Other theorists have proposed differing models to the triple-code, in which there are modules for comprehension, calculation, and number production. The comprehension module translates word and Arabic numbers into abstract internal representations of numbers, calculations are performed on these representations, and then the abstract representations are converted to verbal or Arabic numbers using specific number production modules. Here amodal abstract internal representations of numbers are operated on, rather than numbers represented in specific codes (i.e., quantity, verbal, or Arabic).

The differences between these models are great. For example damage in the first would give rise to failures of one modality of processing or another, while in the second particular abstract operations would be impaired. Functional activation would be different in the two cases when stimuli involving mathematical processing are presented to the subject.

What do all these brain studies add up to and what bearing do they have on the sort of processes that go on in advanced mathematics? Although the subject trials rarely engage anything resembling the sort of advanced mathematics performed at the graduate level, they do suggest that a broad spectrum of brain areas are involved in mathematical reasoning, involving spatial

transformations, visual representation of closeness and relative position on the number line, recognition of numbers and algebraic expressions, making strategic and semantic decisions and transforming many of these processes into coded linguistic transformations as they become familiar and memorized. They also suggest that much of the basis for the richness of mathematics as a palpable reality come from sensory and spatial processes in contrast to the emphasis placed on formal linguistic logic in advanced mathematics.

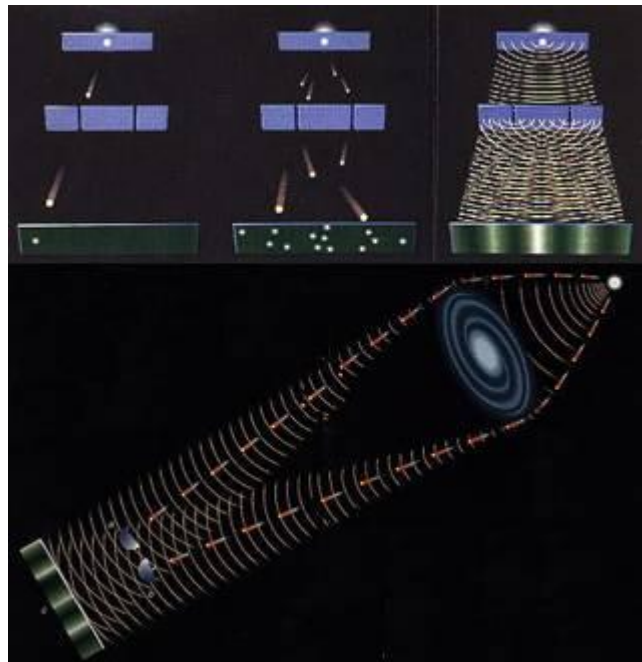
To ensure mathematics continues to be a real part of human culture and doesn't suffer the same fate as classical languages such as Latin in a world of pocket calculators and laptop computers which obviate the need for mathematical expertise in much of the population, mathematicians need to stay in touch with the perceivable richness of science and artistry and imaginative challenge many directly perceivable areas of mathematics do provide without consigning all such problems to the trash can of triviality in an era when new classical results at the research level can only be produced in esoteric spaces through formal processes that stretch far beyond the rich landscapes human imagination into the ivory towers of formalism.

7: The Fractal Topology of Cosmology

An acid test of abstract mathematics as a description of reality is how well it fits naturally with the emerging cosmological description of reality we are in the process of discovering. While physics had to face the demise of the classical paradigm forewarned in Kelvin's two small dark clouds of quantum theory and relativity, classical mathematics has not yet come to terms with these changes to its singular foundations.

Figure 7.1 Top: Quantum interference invokes wave-particle complementarity. Bottom: Wheeler delayed choice experiment.

Quantum reality and cosmological relativity display troubling features which raise questions about the classical model of mathematics based on point-like singular elements in a space whose geometry is independent of its components. Rather than contrasting the discrete and continuous, quantum theory is indivisibly composed of complementary entities which possess both features through wave-particle complementarity, as illustrated in the interference experiment, figure 7.1, in which quanta released as localized 'particles' from individual atoms traverse a double slit as waves, only to be reabsorbed by individual atoms on a photographic plate in the interference fringes. Such complementarity arises from a feedback



process between dynamical energy and wave geometry, as expressed in Einstein's law: $E = \hbar \nu$ [7.1]

However the space-time properties of these quanta are counterintuitive, as can be seen from the Wheeler delayed-choice experiment, where changing the absorbing detector system, from interference detection to individual particle detection can change the apparent path taken by the quanta, long before they arrived.

Worse still, in contrast to quantum theory, which is usually couched within space-time, general relativity applies a second feedback between energy and geometry, in the form of curvature of space-time, so that the geometry and topology of space is also a function of the dynamics. This makes integrating quantum theory and relativity a conceptual nightmare, because, in the event virtual black holes can be created by quantum uncertainty, space-time is locally a seething foam of wormholes, resulting in contradictory descriptions.

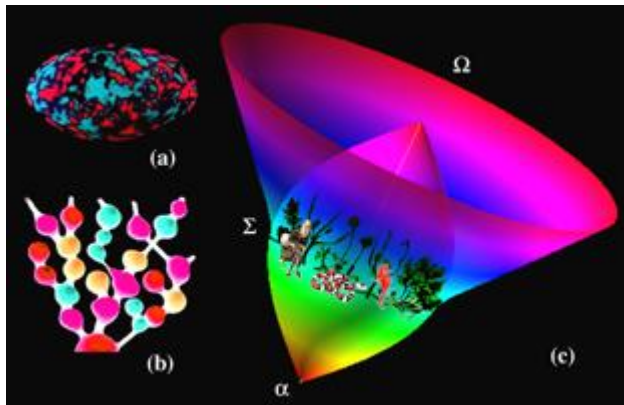


Figure 7:2 The red-shifted cosmic fireball (a) has fluctuations consistent with being inflated quanta. Fractal inflation (b) provides a topological model of how the large-scale structure of the universe might expand forever. Whether or not it does is also a topological question between a closed and open manifold structure.

An oracle for the fit of classical mathematics with reality is the elusive TOE, or theory of everything, which has remained just around the corner since Einstein made inroads into both quantum theory and relativity. In every respect, the search for a unified cosmological theory fundamentally brings topology into the picture and lays siege to classical notions such as point singularities.

Inflation, as a key candidate theory of cosmic emergence, links events at the quantum and cosmological levels. Symmetry-breaking between the forces of nature at the quantum level is coupled to a switching from a phase of cosmic inflation in which an 'anti-gravity' causing an exponential decline in the curvature of the universe switches to attractive gravity, the kinetic energy thus equaling the gravitational potential energy, enabling the universe to be born out of almost nothing.

At the quantum level, theories uniting gravity and the other forces are based on a variety of forms of symmetry-breaking, in which the differences between the two nuclear forces, electromagnetism and gravity arise from symmetry-breaking transformations of a super-force.

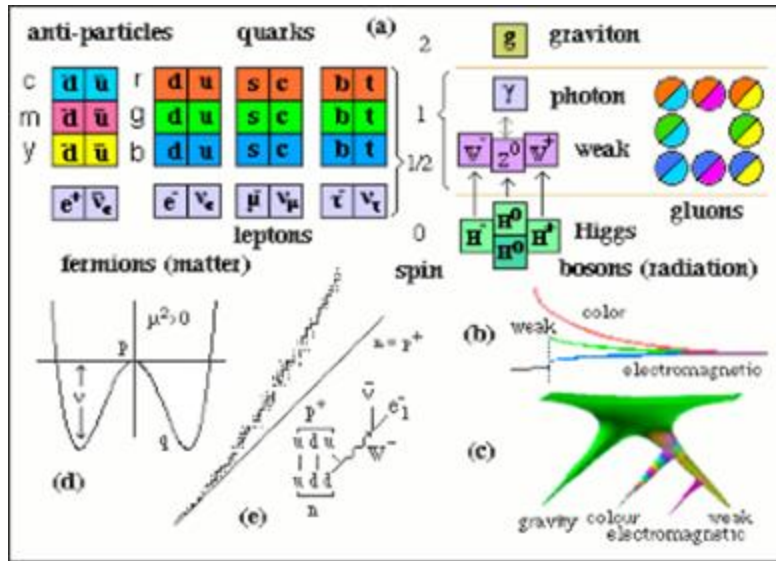


Figure 7.3 The standard model of physics involves a symmetry-breaking between electromagnetism and the weak and colour nuclear forces. A deeper symmetry-breaking is believed to unite gravity with the others.

In the standard model of particle physics, the divergence, first of the weak force from electromagnetism, and then the color force of the quarks and strong nuclear force are mediated by forms of symmetry-breaking in which the bosons carrying the weak force take up a scalar Higgs' particle and thus gain non-zero rest mass, at the same time quenching the inflationary anti-gravity effect of the Higgs' field. The latent energy released by this process gives rise to the hot shower of particles in the big-bang's aftermath. A similar but slightly different symmetry-breaking applies to the colour force that binds quarks, involving massless bound gluons.

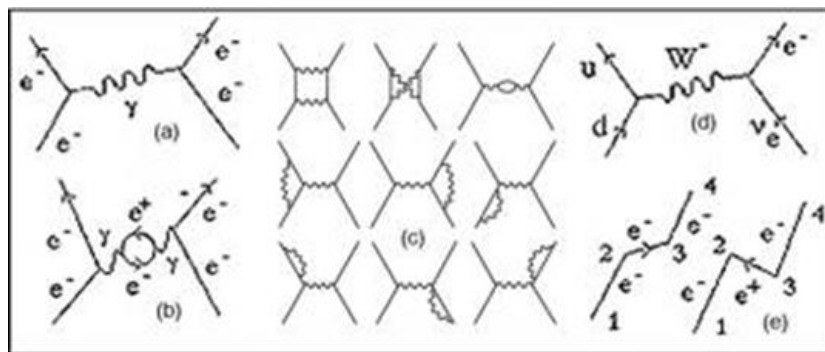


Figure 7.4 Feynman diagrams (a) 2nd order and (b) sample 4th order terms in the infinite series determining the scattering interaction of two electrons. (c) The full set of 4th order terms. (d) The weak W particles act as heavy charged photons indicating symmetry-breaking. (e) Time-reversed electron scattering is positron-electron creation annihilation, showing virtual particles are time reversible^[64].

Quantum field theories are fractal theories, because they define the force, say the electromagnetic scattering between two electrons, in terms of a power series of terms, mediated by virtual photons, summing every possible virtual particle interaction permitted by uncertainty, each of which corresponds to an increasingly elaborate Feynman space-time interaction diagram (figure 7.4 a-c). The series is convergent in the case of electromagnetism because the terms diminish by a factor $\frac{e^2}{hc4\pi\epsilon_0} \sim \frac{1}{137}$ [7.2] the so-called fine-structure constant. A major quest of all theories is such convergence, to avoid infinite energies or probabilities.

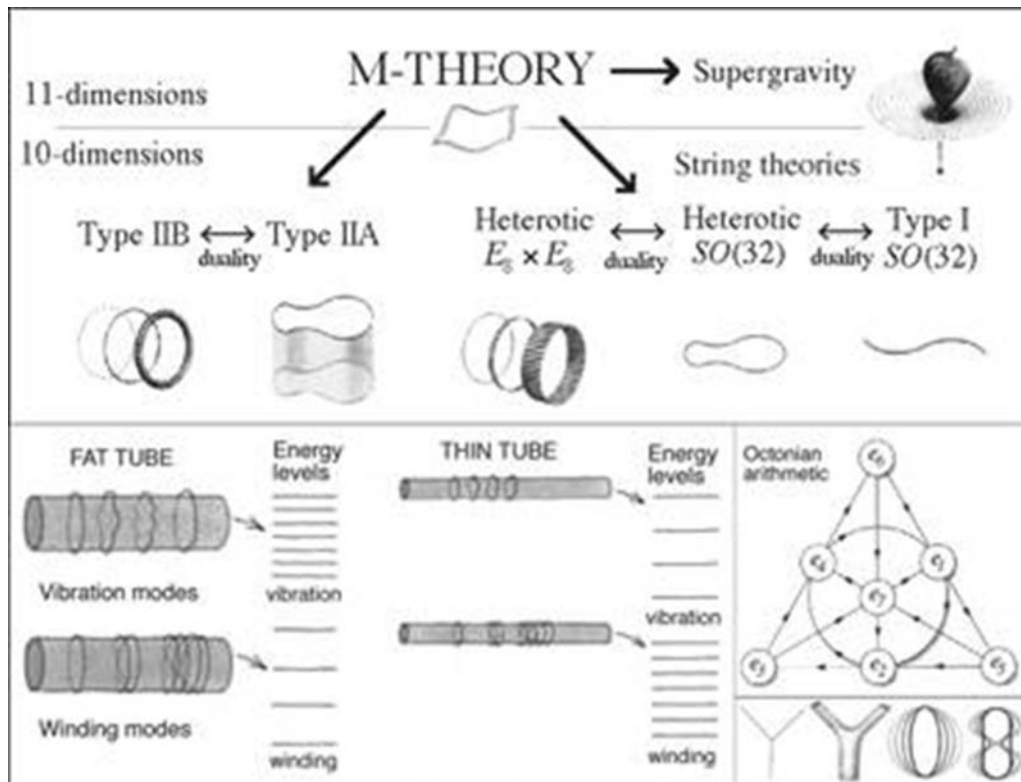


Figure 7.5 Top: M-theory can unite several 10-D string theories and 11-D supergravity through dualities^[65]. The holographic principle allows an n -D theory to be represented on an $(n-1)$ -D surface. Lower left: dualities between theories can exchange vibration and topological winding modes of strings on the compactified dimensions^[66]. The algebra of the groups may invoke the octonians, lower right. String excitations, bottom right, avoid point singularities, but result in topological connections when strings meet.

Attempts to unite gravity with the other forces have proved more difficult, with a series of theories striving to hold the centre ground, from supergravity, through superstring^[67] to higher dimensional (mem)brane M-theories^[68]. All these theories have topological features attempting to get at the root of the singularities associated with the classical notion of point singularity and its infinite energy. They are broadly based on supersymmetry^[69] – the idea that every force carrying boson of integer spin is matched by a matter-forming fermion of half-integer spin to ensure their independent contributions balance to give rise to a convergent theory. All string and brane theories are founded on removing the infinite energy of a point singularity by invoking the

quantum vibrations of a topological loop or string, or membrane for small distance scales, resulting in a series of excited quantum states. Connecting several of these theories are principles of duality in which two theories with differing convergence properties can be seen to be dual, so that a non-convergent description in one corresponds to a convergent description in the other. This can result in dual descriptions of reality in which supposed fundamental particles, like quarks and neutrinos exchange roles with supposed composites of exotic particles like the magnetic monopole singularities of symmetry-breaking. These theories also share a basis in invoking a higher dimensional space, usually of 10 to 12 dimensions to make the theories convergent. This in turn raises the notorious compactification problem of how some of these dimensions can be topologically 'rolled up' into closed loops forming internal spaces representing the 10-12 internal symmetries of the twisted form of the forces of nature we experience as well as the four dimension of space-time. These theories involve topological orbifolds^[70] – orbit generalizations of manifolds factored by a finite group of isometries, Calabi-Yau manifolds^[71], topological bifurcations, and potentially up to 10^{500} candidate string theories^[72] hypothetically representing multiverses with differing properties, only a vanishing few of which would support life and sentient observers, thus invoking the Anthropic principle^[73], rather than cosmologically unique laws of symmetry and symmetry-breaking.

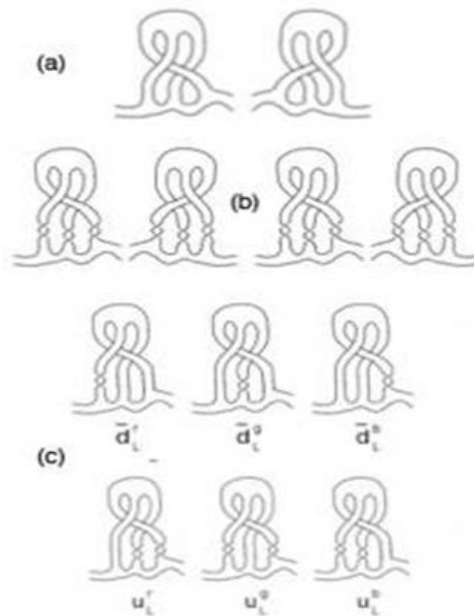


Figure 7.6 How the lightest family of particles in the standard model appear as braids^{[74], [75], [76]}. Each complete twist corresponds to a third unit of electric charge depending on the direction of the twist. (a) Electron neutrino and anti-neutrino correspond to mirror-image braidings. (b) Four states corresponding to the electron and positron with charge depending on the orientations of the twists. (c) Three colours of up quark and anti-down quark.

An alternative to string and brane theory is loop quantum gravity^[77] and topological quantum gravity based on braided preons (figure 7.6). Here again we have a topological basis, in which the fundamental particles are braids in space-time, consisting of more fundamental units called preons, three of which make up each quark and each lepton. The orientation of the twists in these

braids determine a fractional electric charge of $\pm\frac{1}{3}$ which can sum in differing ways to the charge on the electron and positron or up and down quarks. The theory predicts many features of the standard model including the relationship between quarks and leptons, the charges of the two flavours of quark – up and down and the fact that each of these come in three colours corresponding to the combinations of one and two twist braids on the triplet and can model particle interactions through concatenation and splitting of braids.

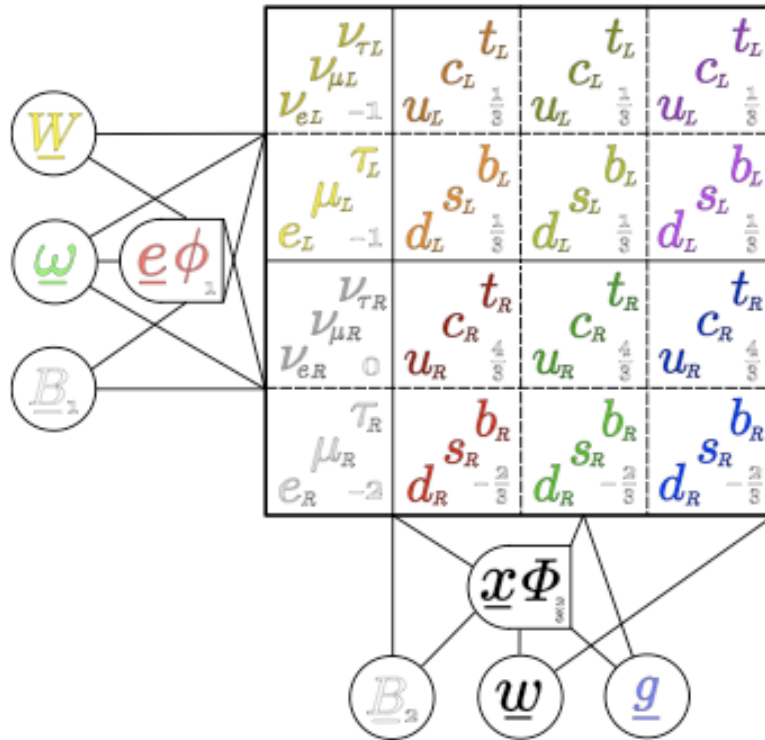


Figure 5: A periodic table of E8.

More recently Garrett Lisi’s “Exceptionally simple theory of Everything” [78], [79] attempts to integrate all the forces including gravity and interactions of both fermions and bosons in terms of the root vector system generating E8, with its subalgebras such as G2 and F4 representing sub-interactions, such as the colour force. The connection he uses again represents the curvature and action on a four-dimensional topological manifold

We thus find that all the candidate theories of reality have an intrinsic topological as well as an algebraic basis and all lead to situations in which the classical view of mathematical spaces is replaced by quantized versions, which fundamentally alter the founding assumptions. One can then ask whether the difficulty at arriving at a theory of everything results from the obtuseness of physicists, or the inadequacy of abstract mathematics as a cultural language of ideals to come to terms with the actual nature of the universe we find ourselves within.

8: References

In the interests of the maze-like nature of mathematics, these references have an emphasis on interlinked internet resources, particularly those from Mathworld and Wikipedia, which themselves provide direct access to the magical maze in the noosphere, which mathematics has become.

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