Article

The Catuskoti

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Abstract

An essential principal to Buddhism is non-dualism. However, the Catuskoti is clearly a system still immersed in dualism. This sort of dualism is more like that of the dual in Tao. Taken by themselves, the two relative states contain within themselves the nature of the absolutes. The only thing which differentiates are the notions *both*, *neither*. It is much like the yin and yang symbol. However, more accurately as we go on we see a fractal emerge. Hence, the ultimate truth, one in which we seem to conceptually call the more subtle truth is an illusion. The infinite recursion of this extension hints at an ultimate truth arising at ∞. The conception which takes within it this very fractal nature is truly enlightened. A truth which is free from dualism is either entirely immersed within dualism, or it lacks the distinction of truth all together. The use of the Catuskoti serves the purpose to hint ultimately at a non-truth. Speaking in terms of tautologies and ineffables, we will see the Catuskoti is a conceptual elaboration of traditional dualism, absolute true and false. While this itself is a conceptual elaboration of the union of true and false, Sunyata or 0. Sunyata is a conceptual elaboration of itself, which of course cannot be explained conceptually because then it emerges from non-conceptual Sunyata to conceptual Sunyata of 0. We can hint at it by saying, as a truth space, the non-conceptual Sunyata be U, then the set of ineffables of U and tautologies of U forms the conceptual elaboration of U. It should be clear that careful attention to our use of V_4, the Klein 4 group, will be sufficient to realize a conceptual grasp of the non-conceptual Sunyata.

1 Building Blocks

1.1 definitions and propositions

Definition 1.1.1. A **Truth Space** U is a collection of objects with a relation, $*$.
For our purposes, the relation will be addition, which behaves in a truth space as exclusive disjunction [2].

Prop 1.1.1. *A **truth space** is an algebraic group.*

Definition 1.1.2. A **truth value** is an object within the truth space U, written u.
Hence U=$\{u_0, u_1, ..., u_n | u_i * \ u_j \in U\}$, the truth space U is equal to the set of objects such that there is a relation defined by $*$. Since the truth space is a group, any relation between two truth values will return a truth value.

Definition 1.1.3. A truth state is a set of truth values, which are treated as a single object. A set of truth states will be applied to **propositions**, truth possessing statements.

Hence, a truth space and a truth value are both truth states, and a truth space can have many truth states.

Prop 1.1.2. *There exist some U which may be constructed from a dualistic view to truth states, which considers absolute and relative truth states.*

The above proposition is one of the linguistic approaches we may take investigating truth. It is purely conceptual, and without a nature of truth there is nothing to be dualistic. Two truth states which form a dualistic view are called **conjugates**.

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[2] This is because all of our truth spaces will be equal to some direct sum of sets of integers.**integers**, **direct sum**.

Definition 1.1.4. A **symmetry** is a **reflection** or **rotation** of a **geometric realization** of a truth space U, which preserves the geometric orientation.

Prop 1.1.3. *The set of symmetries also forms an algebraic group.*

Definition 1.1.5. A **negation** is any map which under the group operation are their own inverse.

Prop 1.1.4. *All reflections, and other 2 cycles of the symmetry group, are negations.*

Definition 1.1.6. A **partial negation** is a symmetry which is not its own inverse. A finite number of iterations of a partial negation forms a negation.

Hence, if we have a truth space with truth values, then taking a symmetry of the geometric realization of the space will 'map' each truth value to another truth value. For a symmetry which is a negation, then applying the symmetry twice returns the original orientation. A rotation by $\pi/2$ radians is not a symmetric negation of the truth space $\mathbb{Z}_2$, since it is not even a symmetry. More so, even if one conceived of rotating by $\pi/2$ then it wouldn't be a negation since $\pi/2 + \pi/2 = \pi$. π is a reflection(or rotation), which is a negation, so it $\pi/2$ is not a negation itself. However, this will not stop us from rotating spaces in non-symmetric orientations-such as rotating the square $\pi/4$ radians.

Definition 1.1.7. An **algebraic negation** is a map from one collection of truth values to another in which the map can be defined by an algebraic operation on each and every truth value, regardless of the geometric realization(or placement on a matrix). In fact, an algebraic negation can be constructed by adding an element of the truth space to each truth state. Equivalently, an algebraic negation may be constructed by adding an element of the set of tautologies to the truth space.

Definition 1.1.8. A **tautology** is a space in which all dualistic constructed views of a set of truth states are equal.[3]
The set of tautologies of U is written $\{\underline{U}\}$

Definition 1.1.9. A proposition whose truth state is not a truth value in the given truth space U is said to be **ineffable** in U if for each truth state it is distinct[4] and symmetric.[5]
The set of ineffables of U is written $\{|U|\}$.

Prop 1.1.5. *Given a truth value v in a truth space V, then we can exponentially expand v with respect to an integer k,* $v^k =< v, v, v, ..., v >$ *k times, for* $k \in \mathbb{Z}$*. The geometric realization or algebraic realization of the space should be preserved in the exponential expansion.*

Definition 1.1.10. An **ineffable negation***(partial symmetry)(internal symmetry)* is a symmetry of a truth space U which assigns to each truth value in U a truth state with more than 1 truth value, by an exponential map proposed above, and then permutes these truth values from one state to another along some path.

Definition 1.1.11. An **Infinitesimal Negation** is an ineffable negation, where each truth state has an infinite number of truth values.

1.2 Examples

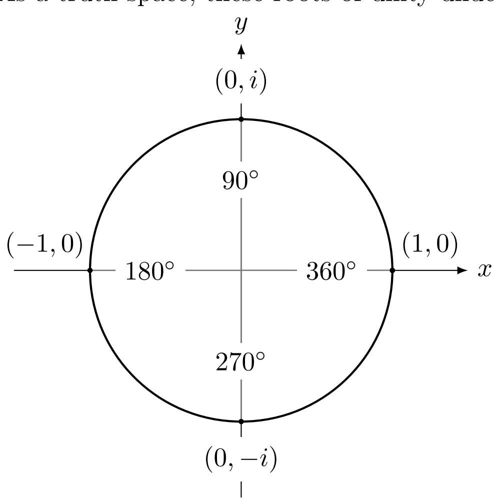
The following examples roughly go in order of the definitions and theorems. However, the most developed example presented is section 2, the Catuskoti. For practice applying the previous definitions and propositions try conceptually elaborating further on some of the following examples.

[3]Conjugate pairs are equal
[4]Distinct here means that it is not an algebraic expression, it is of the same type of truth value as the other states.
[5]Conjugate truth states are conjugate values in some truth space. Equivalently the sum of the conjugate states is equal to some absolutely true tautology

Example 1.2.1. Let U=$\mathbb{Z}_2$, where our group operation is addition modulus 2. This defines a space $\{0, 1|+\}$, where 0=false, and 1=true. Notice, 1+1=2=0 mod 2. This is traditional dualism, and Boolean logic is based of this group.

Example 1.2.2. U could be $\mathbb{Z}_n$, for any n $\in \mathbb{Z}$. Then 0= false, n=true, and all the other values from 1 to (n-1) are intermediate truth values.
For example, if n=3, we have {0,1,2}, and 1='between true and false'. It's not that it is both, or neither, rather it's like a path from 0 to 2, and 1 is just an intermediate value.

Example 1.2.3. Let U be the 4 roots of unity, U=$\{1, i, -i, -1\}$, but the group operation is multiplication. As a truth space, these roots of unity under multiplication are equivalent to $\mathbb{Z}_4$ under addition.



Both these roots of unity and the values of $\mathbb{Z}_4$ can be arranged to form the vertices of the square.

Example 1.2.4. The set of symmetries of $\mathbb{Z}_2$ is $\mathbb{Z}_2$. We can see the symmetries by arranging the group as a line segment with endpoints identified as the truth values. Then, there is a trivial rotation by 0, and a half rotation or reflection such that the line [0,1] goes to [1,0]

$0 \longmapsto 1$

The non trivial negation takes this geometrical realization and 'maps' it to

$1 \longmapsto 0$

Example 1.2.5. Let U, truth space, equal the integers mod 3, $\mathbb{Z}_3$=$\{0, 1, 2\}$. The 3 values can be arranged in a euclidean space either along the same line, or to form vertices of a triangle. If it were arranged as a line, then there would still only be 2 negations, and the intermediate value of 1(between true and false) is an equilibrium point. That is, under any possible symmetry, it is invariant.

Example 1.2.6. For $\mathbb{Z}_2$, all symmetries are negations. Furthermore, there are only 2 ways of algebraically negating the space. Adding either 0 or 1 to each truth value in the space forms the two algebraic negations.

On $\mathbb{Z}_2$ alone, symmetries are algebraic negations. 0 is the trivial rotation, a rotation by 0 radians. 1 is our only reflection. In general the algebraic negation is not always a symmetric negation. In fact, if U is not internally symmetric, then the algebraic negation is not equal to the symmetry of the space. Hence, if we construct a geometric realization of a space, it would be convenient to have an internally symmetric space.

Example 1.2.7. For $\mathbb{Z}_3$, there are at most 3 reflections, hence at most 4 negations (including the trivial negation). There are up to 3 rotations of this space. If there are 3 rotations then the space is geometrically arranged as an equilateral triangle. Then negations can clearly be seen as permutations of vertices. In

fact, all reflections are 2 cycles and all rotations are 3 cycles: a 2 cycle is by definition a negation. However, algebraically negating elements poses a greater difficulty, since the sum of any non-zero element with itself is non-zero: $1 + 1 = 2, 2 + 2 = 4 = 1$ modular 3. Hence, adding an element of the truth space to other elements does not act as a full negation, rather it acts as a partial negation (see definition 1.1.7). Adding any element to the space maps the set of truth values to itself. However, adding a truth value twice does not map each element to itself, hence addition of a truth value is not a full negation. A geometric realization of this space allows us to conceive of some negations, but they're inverting a subspace of $\mathbb{Z}_3$. Hence, there is no negation of the space which permutes each truth state, and iterated twice leaves the identity.
Let's say that the space was arranged as a line, again there is only 1 non-trivial symmetry (from example 5). There is only 1 non-trivial symmetry, which fixes 1, but 1 does not remain invariant under algebraic sums(or partial negations). So, while one could perform any combination of symmetries on this space, one cannot compute a negation of 1 from the symmetries alone.

Example 1.2.8. Let the truth space U=$< 0, 1, 1 >$. This is odd, because the truth value 1 arises twice in two separate truth states. Since U is not internally symmetric, algebraic negations will be different than symmetrical negations. One way of interpreting this is that the space is $\mathbb{Z}_3$, but the states are only filled with these two values. Algebraically negating the space can be done by adding an element of $\mathbb{Z}_3$ to each truth state. Given U=$< 0, 1, 1 >$, U+0=U, U+1=$< 1, 0, 0 >$, U+2=$< 2, 0, 0 >$. Notice if we add an element not in U, but in $\mathbb{Z}_3$, to U the resulting truth space is related to the intersection of the two spaces.

Example 1.2.9. $\mathbb{Z}_3$ has at most 2 partial negations, rotations. If the geometric realization is an equilateral triangle then there are 6 symmetries, 3 rotations(including the trivial symmetry), and 3 reflections. The two non-trivial rotations are partial negations. Assuming there is such a geometric realization for this space, then we can write the space as $< 0, 1, 2 >$
Let any non-trivial rotation be ρ, then $\rho(< 0, 1, 2 >) =< 1, 2, 0 >$, hence,
$\rho(\rho(< 0, 1, 2 >))) = \rho(< 1, 2, 0 >) =< 2, 0, 1 >$. Rotating once more will leave us with the identity.

Example 1.2.10. From example 3, the geometric realization of the truth space is a square, with vertexes identified by truth values. There are a total of 8 symmetries, forming the dihedral group of order 4, we will see this construction again in section 2, constructing the truth space for the Catuskoti.

Example 1.2.11. Consider the space $\mathbb{Z}_2$, then the space is written $< 0, 1 >$, and the ineffable states are $| < 0, 1 > |$ and $| < 1, 0 > |$. To construct an ineffable negation generally would require tautologies of the system, and then permuting the truth values of the tautologies to form ineffable states. However, this space is well behaved, and we can infer that the ineffable negation of the entire space will map each truth value to one of these ineffable states, given only 2 truth values and only 2 negations. Hence, an ineffable negation of $\mathbb{Z}_2 =< 0, 1 >$ is $< | < 0, 1 > |, | < 1, 0 > | >$. Another ineffable negation would be a symmetry of this, and hence there are only two ineffable negations of this degree.

These ineffables are ineffable in all $\mathbb{Z}_n$ for all n.

Example 1.2.12. Given the truth space U=$\mathbb{Z}_4$, 2 and 3 are both intermediate to the extremes of 0 and 1, false and true respectively. 0 and 1 are both absolute truth values, and 2,3 are relative.

Example 1.2.13. Let U=$\mathbb{Z}_2$, there is only 1 diametric set(0 and 1 are perceived dualities), hence only one constructed view of the truth space to form tautologies. Hence, we may write U as a list of tautologies. $U = \{\underline{< 0, 0 >}, \underline{< 1, 1 >}\}$. Notice it is a list of the entire space, not an ineffable.

Example 1.2.14. Infinitesimal Negation: Using $\mathbb{Z}_2$, let $\lim_{k\to+\infty} \underline{0}^k =< 0, 0, ..., 0 >$, with infinite elements, and $\lim_{k\to+\infty} \underline{1}^k =< 1, 1, ..., 1 >$.
Notice if in this infinite expansion each index in order represents a modular 2 placement holder, like a decimal place, then 0=0, but 1 is approximated by $1/2^1 + 1/2^2 + ... + 1/2^n$ and negations are adding infinitesimal numbers. An infinitesimal negation of the space $\{\underline{0}, \underline{1}\}$ is $\{< 0, 0, .., 0, 1 >, < 1, 1, ..., 1, 0 >\}$

1.3 Logical Extension Theorem

If a proposition was described by an ineffable which has the structure of the truth space, it means the behavior of the proposition under a map will behave as though the states were actually values in the truth space. If we add two ineffables then we will add their truth states, and thus their truth values. However, when we interpret the system again, the discrete value which one may actually interpret isn't a truth value, but is an ineffable of the same dimension of the propositions.

If a sentence adds two ineffable symmetries then the sum is equivalent to a sentence with two distinct propositions, v_1, v_2. If one assigns a truth value to v_1 or v_2, it does not describe the behavior of v_2 or v_1 and the result is a set of truth values. One would have to assign a value to both virtual propositions to have a result which is a truth value.

Theorem 1.3.1. *Given a truth space* U_0*, then there exists an* ***extension*** *of* U_0 *,* $E : U_0 \to U_1$ *where* $U_1 = \{|U_0|\} \cup \{\underline{U_0}\}$*.* U_1 *is the union,* $\cup$*, of the set of ineffables with the set of tautologies.*

Lemma 1.3.2. *Given a truth space* U_k*, using the above extension, we may extend towards infinity, and retract to a truth space with no truth values.*

Theorem 1.3.3. *If* $a,b \in |U|$*, then* $a+b \in \underline{U}$*. In fact a+b is an object describing a phase of a+b. This is because tautologies can be negations, hence the phase difference is what angle or line of symmetry relates a and b.*

2 Building the Catuskoti as a truth space

The Catuskoti is a method of logic [6] [7]. Even though it is not the entire theory that we will call V_4, V_4 is easily apprehended linguistically by the Catuskoti[8]. In general, the Catuskoti considers a single proposition, which in itself is composed of known propositions, perhaps more accurately considered dharmas[9], since they are atomic and essentially uniform. Furthermore these dharmas, the atomic elements of a proposition, usually in essence are ineffable themselves.

Example 2.0.1. Brahmic philosophers worded a question to the Buddha fully aware of the general nature of the truth space, connectives, and negations.[10] The question is based off of the proposition, 'The arhat exists after death'. The arhat is an enlightened person, but not just any enlightened person. The arhat is like a flame.[11]

'How is it, Gautama? Does Gautama hold that the arhat exists after death, and that this view alone is true, and every other false?

The Buddha and Vaccha go through all 4 possibilities of this proposition in V_4, and each one is rejected when Vaccha proposes that specific orientation is true. Does the arhat exist after death, does the arhat not exist after death, does the arhat exist and not exist after death, does the arhat neither exist nor not exist after death. When they have exhausted all 4, Vaccha is confused, he expected some sort of true value. That is, he expected to find that the arhat had some true nature after death, or at the very least know his question was dialectically different. To Vaccha's perspective, when the Buddha rejects the absolute duality, the nature of the enlightened being must be a type of relative value, but these too are

[6] Is it deductive logic, is it inductive logic, is it both or neither?
[7] A category which constructs a logic
[8] Mathematics presupposes any conceptual nature of emptiness which is conceptually elaborated by the Catuskoti itself
[9] A dharma is a philosophical element in Buddhist philosophy which contains within itself its entire nature. That is, it does not borrow its nature from any other dharma or from any compounded substance or non-substance.
[10] That is the following truth space V_4.
[11] There are numerous metaphors which explain the conceptual nature of an arhat

rejected. Therefore, Vaccha is left without any sort of orientation, the answer to his question is not a value in his logical system.[12] Unknowingly perhaps, his views are still dualistic, and cannot in themselves get to the essential nature of the arhat. Therefore, it would seem that the nature of the arhat is ineffable, the truth value is not a truth value in V_4. In what sense is it ineffable? Is it locally ineffable, or is it globally ineffable? That is, is it ineffable only in V_4 or some finite extension of V_4, or is it ineffable in all possible extensions of V_4?

We will first attempt to yield the same result he expected. To do this, however, we will use both exclusive disjunction and inclusive disjunction on our binary parts of the sentence. These disjunctions are ways of saying "or". One reason for this is because the Brahmic philosopher does not seem to ask other negations,[13] principally the partial negation of ρ, the $\pi/2$ rotation. Though even if it were asked as a dual pair, it would be simultaneously rejected. The Brahman expects that if all statements are asked then they will not all be true, and they will not all be false. Hence, he would most likely consider exclusive disjunction, and that's essentially how the dialogue appears. The Buddha will show that the nature of the Arhat is neither nihilistic nor eternalistic. Either interpretation necessitates the assumption that the premises are all valid, however as Vaccha asks each, they are simultaneously rejected. However, this negation does not reject all inclusive disjunctive interpretations since the question added that all other views were false. If this part of the sentence were left out, one could interpret the system using inclusive disjunction, and then could look for the possibility that two states are both valid. However, as we will see because of the Buddha's response, all possible combinations would be rejected.
We will then explain an interpretation of the Buddha's position. Before we can do either, we need a truth space, negations, and 3 connectives.

An essential principal to Buddhism is non-dualism. However, the Catuskoti is clearly a system still immersed in dualism. This sort of dualism is more like that of the dual in Tao. Taken by themselves, the two relative states contain within themselves the nature of the absolutes. The only thing which differentiates is the notion of *both* or *neither*. [14]. It is much like the yin and yang symbol. However, more accurately as we go on we see a fractal emerge.[15] The infinite recursion of this extension hints at an ultimate truth arising at ∞. As we approach infinity we arrive at more subtle truths, which conceptually form an illusion. A truth which is free from dualism is either entirely immersed within dualism, or it lacks the distinction of truth all together. The use of the Catuskoti serves the purpose to hint ultimately at a non-truth[16]. Speaking in terms of tautologies and ineffables, we will see the Catuskoti is a conceptual elaboration of traditional dualism, absolute true and false. This itself is a conceptual elaboration of the union of true and false, Sunyata or 0[17]. Sunyata is a conceptual elaboration of itself, which of course cannot be explained conceptually because then it emerges from non-conceptual Sunyata to conceptual Sunyata of 0. We can hint at it by saying that as a truth space, the non-conceptual Sunyata is labeled by U, then the set of ineffables of U and tautologies of U forms the conceptual elaboration of U.
It should be clear that careful attention to our use of V_4 will be sufficient to realize a conceptual grasp of the non-conceptual Sunyata.

[12] It would have been interesting to read the discourse immersing into partial negations, or infinitesimal negations to find truth. Hence, they would consider specific ineffables which are constructed from unifying conjugate pairs, for which there is only 1 to consider, and it is called ρ. Being that this doesn't arise, one would conclude the Brahmic philosophers themselves did not conceptually elaborate upon other negations.

[13] Which would extend their conception beyond the four truth states.

[14] Imagine if we were to say some of true, or some of false. Then we might say, z=some true and some false. Though, we're always in a mathematical dualism created by our conjugate pairs, so there is $\overline{z}$. Furthermore z or $\overline{z}$ is tautologously true. It is in this disjunction that 'true' always arises again, and we see it embedded within these truth values.

[15] If the first error of dualistic thinking is a circle with one side being black the other white{*one is 0 the other is 1, one is false the other is true*}, the second is the Tao,{*x and y*}, perhaps the next could be viewed as ρ and $\overline{\rho}$. Which have within themselves x and y.

[16] Not to be confused with a non-true a negation of a perceived absolute true, or even a negation of a perceived absolute truth. It is a non-truth of any nature.

[17] This conclusion is taken from many authors perception on the relationship between mathematics and Buddhism.

Example 2.0.2. If Atman[18] were some sort of permanent dharma, which existed everywhere undifferentiated and equally, then upon the thought of the conjunction of two things as far as the mind extends there should equally be that which was called Atman. However, all that is eventually found is emptiness of 0. The conjunction of an infinite series of aggregates, which may appear different than Atman, will leave only the essential nature of Atman, since it is assumed that Atman will permeate everywhere the mind extends. One will come to see, because of x and y, the conjunction of this infinite series leaves 0, emptiness, unless everything is equal to one non-zero truth value. We come to a reinterpretation of an empty Atman which is ineffable{in V_4}. For some sentence to be valid from this, all states must be equal upon further constructions.

Theorem 2.0.4. *Let U_2=$\mathbb{Z}_2$, a Boolean logic composed of only true and false equal to 1 and 0 respectively with respect to addition. Then from theorem 1.1.1 we can define* $U_1=\{| < 1,0 > |, | < 0,1 > |\} \cup \{\underline{< 0,0 >}, \underline{< 1,1 >}\} = V_4 = \{x, y, 0, 1 | \forall a \in V_4, a + a = 0, x + y = 1\}$

Lemma 2.0.5. *U_3 is the Klein group of order 4, V_4. It also represents the truth space for the Catuskoti, just as $\mathbb{Z}_2$ represented the truth space for Boolean logic.*

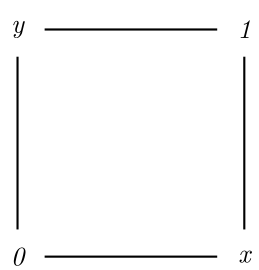
Also, see examples, 1.1.11 and 1.1.13.

Prop 2.0.1. *Linguistically:*
0 is absolutely false asatya
1 is tabsolutely true - satya
x is neither true nor false - asamvrti
y is both true and false - samvrti
Below are two tables which show the connectives of exclusive disjunction(+) and conjunction($\odot$) on all combinations of truth values.

+	0	x	1	y
0	0	x	1	y
x	x	0	y	1
1	1	y	0	x
y	y	1	x	0

$\odot$	0	x	1	y
0	0	0	0	0
x	0	x	x	0
1	0	x	1	y
y	0	0	y	y

Prop 2.0.2. ***Geometric Realization of*** V_4
(i) We may notice we can write V_4 as vertices of the square since from theorem 2.0.4 we notice they look like Cartesian coordinates which we join to form a square. x=$< 1, 0 >$ and y=$< 0, 1 >$.



(ii)We can also identify the edges with arrows to designate topological paths, as to relate this square to the path diagram for the topological space which represents the Klein bottle. This designation can be done in many ways which are all equivalent. There are two paths, the x and y paths, and the arrows can be oriented by the relation $yxy^{-1}x$.

The above orientation is our standard orientation since it is similar to Cartesian coordinates, and it is called e_0.

[18]Ultimate self, some practices suggest realizing that Atman is essentially God.

Prop 2.0.3. *The group of symmetries of the geometric realization of* V_4 *is formed by the group* D_4*, where* $D_4 = \{\underline{0}, \underline{\rho}, \underline{1}, \underline{\overline{\rho}}, \underline{x}, \underline{y}, \underline{\tau}, \underline{\sigma}\}$*. Linguistically:*
0 the trivial symmetry, this is a basis for our standard orientation.
ρ *is a quarter rotation counter clockwise*
1 is a half rotation
$\overline{\rho}$ *is a quarter rotation clockwise*
x *is a reflection horizontally, thus it has a vertical axis of symmetry*
y *is a reflection vertically, hence it has a horizontal axis of symmetry*
τ *is a reflection which leaves 1 and 0 fixed*
σ *is a reflection which leaves* x *and* y *fixed.*

τ and σ are especially interesting because they leave either an absolute (0 or 1, false or true) diagonal or a relative(x or y) fixed while reflecting the other. From this perspective, if we limited our view only to the absolute so we're essentially within a Boolean logic, then τ behaves as 0 and σ behaves as the 1 negation.

Prop 2.0.4. *The set of* ***symmetries of*** V_4*,* $\{|V_4|\}$*, is written* $\{|V_4|\}=\{e_0, e_x, e_y, e_1, e_\rho, e_{\overline{\rho}}, e_\tau, e_\sigma\}$*. As matrices:*
$e_0 = \begin{bmatrix} y & 1 \\ 0 & x \end{bmatrix}$*, This is our standard orientation. Without a choice of a standard orientation we will always be at a loss to interpreting any result of an argument.*
$e_x = \begin{bmatrix} 1 & y \\ x & 0 \end{bmatrix}$, $e_y = \begin{bmatrix} 0 & x \\ y & 1 \end{bmatrix}$, $e_1 = \begin{bmatrix} x & 0 \\ 1 & y \end{bmatrix}$

$$e_\rho = \begin{bmatrix} 1 & x \\ y & 0 \end{bmatrix}, e_{\overline{\rho}} = \begin{bmatrix} 0 & y \\ x & 1 \end{bmatrix}, e_\tau = \begin{bmatrix} x & 1 \\ 0 & y \end{bmatrix}, e_\sigma = \begin{bmatrix} y & 0 \\ 1 & x \end{bmatrix}$$

Prop 2.0.5. D_4 *may be divided into its negations, and partial negations.*
Pure negations: $\{\underline{0}, \underline{1}, \underline{x}, \underline{y}, \underline{\tau}, \underline{\sigma}\}$
Partial negations: $\{\underline{\overline{\rho}}, \underline{\rho}\}$
However, pure negations are not pure tautologies in an algebraic sense. Hence the only pure tautologies are those which relate to V_4*,* $\underline{0}$*,* $\underline{x}$*,* $\underline{y}$*,* $\underline{1}$*.*

One might say a pure tautology is one in which all truth values are the same. Hence τ and σ are not pure negations in V_4, but act as pure negations for Z_2.
We have constructed a geometric realization of the space, considered the set of tautologies, which under exclusive disjunction and a proposition represented by a symmetry of $|V_4|$ acts as a negation to the space. To construct sentences or arguments from these blocks requires the study of maps from one space to another.

There must be parts or objects to our sentence; these objects are either our propositions represented by a symmetry of $|V_4|$ or they are tautologous or ineffable objects. An ineffable acts in many respects as its own proposition, hence we may confine our attention to maps using symmetries of $|V_4|$ and elements of $\underline{D_4}$ as tautologies or negations.

3 Maps

Negations and symmetries are maps of truth values to other truth values. In general, a map does not have to map a truth value to a truth value in the space. A map from e_0 to e_1 maps each truth value to a distinct truth value in V_4, hence the map is 1-to-1. We can consider compositions of relations with these symmetries and negations; this is a further classification of maps and relations. If we continue extending the logic, as we did from Z_2 to V_4, there will continue to arise many nested relations; in fact the degree in which these relations are interrelated goes to infinity. However, underlying these following definitions and propositions on maps is a much greater theory: Category theory. Although we will not formally go into this right now, further revisions will attempt to bring the notion of category theory in conjunction with the Catuskoti into a clearer light.

Definition 3.0.1. A map has a domain and a range, such that the domain is composed of truth values arranged in some truth state or space. The range is the set of truth states given by the map along with the given domain.
Each truth state in the space is said to be mapped to another truth state in the range.

Negations and symmetries are the simplest maps we will use. They are in fact functions. However, one must *connect* a symmetry(or negation) with a truth space(a symmetry of $|V_4|$). We will introduce our connectives completely in the next section; however, we will concern ourselves with two algebraic connectives which will relate negations to a truth space. We have actually seen this already in the previous section when we presented the set of symmetries of the space. These two connectives are exclusive disjunction as group addition, and conjunction which behaves as multiplication. Applying an element of D_4 to a truth space via one of these connectives is called a negation, and is very well behaved.

Definition 3.0.2. A map which is 1-to-1 and onto is a **function** from a truth value to another truth value. All standard symmetries are functions.

Definition 3.0.3. Absolute truth states are along the diagonal with 0 and 1 in e_0. This may be called the main diagonal. Relative truth states are along the diagonal with x,y in e_0. Recall from section 1, we called these conjugates. Below conjugates are defined in terms of negations.

Theorem 3.0.6. ***Exclusive disjunction negation***: *Given* $e_0 \in |V_4|$ *and* $n \in \underline{D_4}$, *then* $e_0+n=e_n=n+e_0$.
Given $n,m \in \underline{D_4}$, $n+m \in \underline{D_4}$, *and if* $n+m=1$ *then* $m=\overline{n}$ *the* ***conjugate*** *of n.*
$e_n + e_{\overline{n}} = \underline{1}$, *and* $e_n e_{\overline{n}} = \underline{0}$.
(i) $u,v \in \{|V_4|\}, u+v \in \{\underline{V_4}\}$
$u = |V_4|_n, v = |V_4|_m, u+v = (\underline{V_4})_{n+m} = \underline{n+m} = \underline{l}, l \in D_4$

(ii)$u,v \in \{\underline{V_4}\}, u+v \in \{\underline{V_4}\}$
$u = \underline{n}, v = \underline{m}, n, m \in D_4,$ *then* $u+v = \underline{n+m} = \underline{l}, l \in D_4$

(iii) $u \in \{\underline{V_4}\}, v \in \{|V_4|\}, u+v \in \{|V_4|\}$
$u = |V_4|_n, v = \underline{m}, n, m \in D_4, u+v = |V_4|_{n+m} = |V_4|_l$
Hence tautologies and ineffables have an interdependent, symmetrical relationship. The sum of an ineffable and a tautology acts both as an algebraic negation, and as a symmetry.

Periodically the term duality, or dichotomy, may be used. When used with respect to the logic, these are conjugate pairs. A dualistic form of thinking does not apprehend on one form of dualism over another, but is in essence always oriented with respect to conjugate pairs. Hence, our entire perspective through this paper is dualistic. Given a sufficient permutation of dualistic views there arises an awareness of non-dualism. However, it seems unlikely to communicate in some non-dualistic sense.

Definition 3.0.4. Conjugate pairs are dual forms of perception: *dualistic views.*

Theorem 3.0.7. ***Conjunction Negation***: *Given* $e_0 \in |V_4|$ *and* $n \in \underline{D_4}$, *then* $e_0 n=e^n=ne_0$.
Given $n,m \in \underline{D_4}$, $nm \in \underline{\hat{D}_4}$. $e^n + e^{\overline{n}} = e_0^1 = e_0$, *and* $e_n e_{\overline{n}} = 0$. $e^n e^m = nme_0 e_0 = nme_0 = e^{nm}$. *If* $m = \bar{n}$, *then* $e^n e^m = e^n e^{\bar{n}} = 0$.

This exponential notation was used in example 14, with respect to infinitesimal negations. This may be slightly confusing, given the exponent is an element of the truth space then it's a conjunctive negation. If the exponent was an integer then it behaves as the example, increasing the dimension of the element with copies of itself. Hence, raising a truth value to a number equal to the number of elements in a certain block structure is a way of constructing tautologies or certain fractal structures. $1^4 = \underline{1} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$.

Definition 3.0.5. $\underline{\hat{D_4}}$=$\{nm|n, m \in \underline{D_4}\} = \{\underline{0}, \underline{x}, \underline{1}, \underline{y}, \underline{\rho}, \underline{\overline{\rho}}, \underline{\tau}, \underline{\sigma}, \underline{\tau^x}, \underline{\tau^y}, \underline{\sigma^x}, \underline{\sigma^y}\}$.
$\underline{\tau^x} = \begin{bmatrix} x & 0 \\ 0 & x \end{bmatrix}$=x$\tau$=$\tau$x, $\underline{\tau^y} = \begin{bmatrix} y & 0 \\ 0 & y \end{bmatrix}$=y$\tau$=$\tau$y, $\underline{\sigma^x} = \begin{bmatrix} 0 & x \\ x & 0 \end{bmatrix}$=x$\sigma$=$\sigma$x, $\underline{\sigma^y} = \begin{bmatrix} 0 & y \\ y & 0 \end{bmatrix}$=y$\sigma$=$\sigma$y
$\underline{\hat{D_4}}$ is the set of conjugate negation elements which is closed with respect to multiplication.

Example 3.0.3. $e_x + \underline{x} = \begin{bmatrix} 1 & y \\ x & 0 \end{bmatrix} + \begin{bmatrix} x & x \\ x & x \end{bmatrix} = \begin{bmatrix} y & 1 \\ 0 & x \end{bmatrix} = e_0$
Looking at the indices we can see clearly x+x=0. On the left hand side we add the x from e_x, and the x from the negation giving 0, hence we have a 0 symmetry of our standard space. We started with an x symmetry and then negated it.

Example 3.0.4. $e_x + e_\tau = \begin{bmatrix} 1 & y \\ x & 0 \end{bmatrix} + \begin{bmatrix} x & 1 \\ 0 & y \end{bmatrix} = \begin{bmatrix} y & x \\ x & y \end{bmatrix} = \underline{\overline{\rho}}$
$x + \tau = \overline{\rho}$. It can be seen that τ is the composition of the rotation counter clockwise and then the reflection over x, so the order in which we compose symmetries is very important. Had we constructed x and then the rotation, it would have been σ not τ which we added to x. $\tau = \rho x = y\rho$ $\sigma = \rho y = x\rho$. $x\tau = x\rho x = \rho y x = \rho 1 = \overline{\rho}$. These are not negations being multiplied, rather it's an algebraic way of writing the geometric relationship. For example, xy is reflected along x and then y, hence our composition is 1-true. Had it been multiplied, we would have 0.

Example 3.0.5. $\underline{\tau} + \underline{\sigma} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \underline{1}$. Notice σ and τ are conjugates.

We can use maps to define sequences and series. Recall example 1.1.14: in this example we used an infinitesimal symmetry on a Boolean logic to create a decimal expansion. Hence, sequences can be used to construct numbers. In example 1.1.14 we may construct any rational number, and approximate many others.

Definition 3.0.6. A sequence of V_4, S_n=$\{v_1, v_2, ..., v_n\}\forall v_i \in V_4$.

Permutations are cyclic sequences, hence 3.0.15 is a permutation of e_0 and defines a sequence.

Example 3.0.6. Any symmetry is a function of some state e_0. Consider $\phi : e_0 \to (e_0)_{\rho^n}, n = 0...k$, then each truth value follows the cycle 0,y,1,x,0,y,1,x....
Furthermore, we can see the cycle starting with e_0, $\{e_0, e_\rho, e_1, e_{\overline{\rho}}, e_0\}$.

Now that there is a defined space and maps on the space, we can begin to linguistically describe these systems. This comes down to how we assign states of truth spaces to a proposition, our choice of maps, and finally interpreting results.

Prop 3.0.6. *A propositional statement may be represented by an element of* $\{|V_4|\}$*, the set of symmetries of* V_4*, as either a space or as an ineffable.*

While in Boolean logic one only has to consider the position of one truth value to know the choice of a state, here we must know 2 positions, which are not conjugate pairs. Since every symmetry of V_4 has conjugate pairs along diagonals, if we know one horizontal or vertical portion of the matrix then we know exactly which matrix it is. For example, if we knew the left hand side of e_0, then we know y and 0 (top down). There is no other matrix which has that specific configuration, which represents an ineffable object, or symmetry of the geometric realization of the truth space.
So, linguistically there is quite the challenge when we wish to chose a state for a proposition. There is

ultimately nothing which stops us from always assigning e_0 as our state unless there is more than one proposition. Then the orientation of different propositions is very important. If a different orientation is desired, it is through the disjunction with n, a negation, or conjunction with some n that we have a different orientation.

The four states may only be investigated through the eight lenses of the Catuskoti, the method of using the eight symmetries, and a discriminative mind which separates one state from the other. The discriminative mind is perhaps the most important part of the entire process: it is the observer principal in many cases.[19] When we refer to a proposition being ineffable, it means the values are not in V_4. However, saying a proposition is represented by an ineffable does not mean the proposition too is ineffable. These subtle differences allow one to interpret results, but also show that without the observer there is no conclusion. If there is no observer then there is no way of interpreting ineffable results.

Prop 3.0.7. *A propositional statement may also be represented by an element of* $\underline{V_4}$.

This means the proposition is a tautology. While it may be a tautology, it still has 4 states, and since not all tautologies have the same state for every truth value, one must still be aware of the states themselves. Notice, it shouldn't matter if a proposition was represented by 1, 0, x or y; it's always true. But if it were represented by $\rho = \begin{bmatrix} x & y \\ y & x \end{bmatrix}$, then some truth states are not equal. Specifically the absolute diagonal is y and the relative is x.

Theorem 3.0.8. *All algebraic negations of truth spaces represented by elements of* $\underline{U_4}$*(the extension of* V_4*) are equal to some symmetry of the space. The algebraic negation of a composition of propositions will not necessarily be equal to a single symmetry.*

Theorem 3.0.9. *Given a matrix whose elements are truth values of* V_4*, and has been constructed from a series of connectives and negations, then if there exists an algebraic negation, there also exists a series of symmetries on the propositional elements of the space and a series of connectives which is equal to the negation.*

This theorem is difficult to grasp. It suggests that if our argument is constructed purely from the symmetry group of V_4 and standard connectives, then the negation of the sentence can be written as a series of symmetries of $|V_4|$ connected through a list of negations and connectives. Important examples of this theorem are expanded upon in the equivalences section [20].
Two other aspects of the truth space which are important to remember are that V_4, Klein Four group, is **abelian**, meaning adding or multiplying two elements does not depend on the order that we add them. This is important because the extension of V_4 forms U_4, a non-abelian group. This commutativity of V_4 has a direct relationship to our ability to have a linguistic grasp of the system, or to even perceive the arising of the systems in a psycho-physical universe. The second aspect to be aware of is absolute and relative diagonals, the absolute being the one with 0 and 1 in e_0, and the relative is the diagonal with x and y in e_0. These are preserved under symmetries, hence the absolute diagonal in e_x has x and y in it.

3.1 Arguments

Definition 3.1.1. An atomic sentence has one proposition and any number of symmetries or maps on it.
A molecular sentence has more than 1 proposition and a series of maps.

[19] Consider observing a quantum system, the very nature of observing it has a direct effect on the state of the system. Much in the same way, our conceptual elaboration of anything within this field of observation has a direct effect on how it will be perceived. That is, none of this arises naturally without the discriminative mind observing the system.

[20] If, however, a more transcendental function were used, perhaps a piecewise dynamical system on a number of propositions whose initial conditions are elements of the symmetry group, then a global negation may not necessarily be so easily apprehended.

Definition 3.1.2. An argument is constructed of one or more sentences, each composed of propositions. An argument has inputs(propositions), and an output(conclusion), a statement to prove.

Prop 3.1.1. *If through an argument one concludes* $\underline{n}$ *for* $n \in D_4$ *then it is a tautology.*

Prop 3.1.2. *If through an argument one concludes p and* p_n*, for some proposition p(which can be atomic or molecular) and* $n \in D_4$*, then one may conclude* $pp_n = p^{\bar{n}}$.

Definition 3.1.3. Pertaining to the above proposition, if n were 1 then it is a tautologous false, since $PP_1 = \begin{bmatrix} y & 1 \\ 0 & x \end{bmatrix} \begin{bmatrix} x & 0 \\ 1 & y \end{bmatrix} = \underline{0}$.
If n=0 then we have our conclusion p, since $PP_0 = P$.
If n=σ then consider $\tau e_0 = \begin{bmatrix} y & 0 \\ 0 & x \end{bmatrix}$: this is relatively valid, but absolutely invalid.
If n=τ then our statement is absolutely valid, but relatively invalid.

Proposition 3.1.2, and definition 3.1.3 allow us to construct indirect proofs.

Prop 3.1.3. *An indirect proof is constructed from a set* $\{A_i\}$*, a set of assumptions, a statement to show P, and the indirect assumption* P_n*. Then along the proof there is some* $(A_j)_m$*. We can deduce from* $(A_j)_m$ *and* A_j*:* $(A_j)(A_j)_m = A_j^{\bar{m}}$[21]
Next we can conclude $P_n(A_j^{\overline{m}})$ *is valid or P is valid, or they're both valid, or neither.*[22]*. It does not say that validity is exactly the same as absolutely true, but that there is a type of validity which is simultaneously interpreted as absolutely true(this is based on what our expectation or our center of perception is). Whichever way, we conclude algebraically* $P + P_n(A_j^{\overline{m}})$.

It may turn out $P + P_n(A_j^{\overline{m}})$ cancels out leaving just P, or it may somehow leave just $P_n(A_j^{\overline{m}})$, or it could leave some combination of the two. Perhaps the solution is a tautologous false: then our interpretation of validity is neither true nor false. If it were 1 then it would be both true and false(again pertaining to validity).

4 Connectives

Definition 4.0.4. Given two propositions A, B which can be represented by the space V_4, we may relate A,B using a connective $*$, such that $A * B = \begin{bmatrix} y & 1 \\ 0 & x \end{bmatrix} * \begin{bmatrix} y & 1 \\ 0 & x \end{bmatrix} = \begin{bmatrix} y * \begin{bmatrix} y & 1 \\ 0 & x \end{bmatrix} & 1 * \begin{bmatrix} y & 1 \\ 0 & x \end{bmatrix} \\ 0 * \begin{bmatrix} y & 1 \\ 0 & x \end{bmatrix} & x * \begin{bmatrix} y & 1 \\ 0 & x \end{bmatrix} \end{bmatrix}$ Where the 4 arrays inside the larger array is B while the 4 truth values written to the left of each array correspond to the states of A.

In general, we will consider $A * B =$

$$\left[\begin{array}{cc|cc} (y,y) & (y,1) & (1,y) & (1,1) \\ (y,0) & (y,x) & (1,0) & (1,x) \\ \hline (0,y) & (0,1) & (x,y) & (x,1) \\ (0,0) & (0,x) & (x,0) & (x,x) \end{array}\right]$$

Such that a*b=(a,b) for a and b $\in V_4$. The horizontal and vertical bars are meant to guide the reader to see the truth states of A. The four quadrants are the 4 possible states of A. Consider then that B is ineffable, then a proposition composed of A and B will yield an ineffable in that quadrant. Specifically each

[21] Notice if m is 1 then $\overline{m}$ is 0 and we have a tautology of 0, and our usual form of contradiction from Boolean logic. If m=0 then we have A_j.
[22] It should be clear that even our interpretations, a statement of validity, is also dependent on V_4. In Boolean logic validity was dependent on the truth space $\mathbb{Z}_2$

block is the truth state of the resulting sentence since individual truth values may not be discriminated against for an ineffable.Depending on the connective, the resulting set of ineffables are not necessarily in U_4.[23]

The above matrix is read such that the largest block form refers to A, and the smaller blocks are B. Furthermore, the first coordinate is A the second coordinate is B. If there were three propositions, the largest block is A, and the smallest is C. The blocks are always read oriented as e_0:0 is in the lower left, x to the lower right, and conjugates along diagonals. If one maintains the view of the Cartesian plane, there will be no difficulty naturally orienting oneself.

If A and B were symmetries of the same proposition, then we would not have a 4x4, we would do calculations as we did for maps on one proposition.

Definition 4.0.5. Exclusive Disjunction: An exclusively disjunctive sentence is false whenever the disjunctives share the same truth value.

A+B= $\begin{bmatrix} 0 & x & x & 0 \\ y & 1 & 1 & y \\ y & 1 & 1 & y \\ 0 & x & x & 0 \end{bmatrix}$ Exclusive disjunction is addition modulus 2 over Z_2 in Boolean logic. It is group addition here too. We've seen this connective through the development of the paper. Notice along every symmetry it is internally symmetric. Along all geometric symmetries there is a corresponding algebraic symmetry. We use exclusive disjunction when we consider A or B, but we don't want the case when their truth values are identical to be true.

Definition 4.0.6. Conjunction, A and B is false whenever A and B have conjugate truth values. A and B is true when the truth value of A and B are both identical to 1. Conjunction is group multiplication.

AB= $\begin{bmatrix} y & y & y & 1 \\ 0 & 0 & 0 & x \\ 0 & 0 & 0 & x \\ 0 & 0 & 0 & x \end{bmatrix}$

Example 4.0.1. A particle P is moving at velocity, v in meters per second, <u>and</u> is at some location z in a measured space. <u>And</u> is our connective, and our two propositions are velocity and position.
If this sentence refers to observing this behavior of a particle, then from the uncertainty principle there exists a probability matrix C such that C is equal to the multiple of C_1 and C_2, the probabilities of measuring an expected velocity or position respectively. However we know $C_1+C_2 \leq 1$. Since we're multiplying the two propositions and we can write C_2 in terms of C_1, then there is a closed form for the sentence, C multiplied by AB. This is not so easily done if we're adding the two propositions. If the observation was true, the expected position and velocity were both measured, then one reads the upper right corner of the matrix AB. Since it is also being multiplied by the probability C, there is a certain probability this observation will be measured. The probability of this event is $\leq a - a^2, a \leq 1$. If a=0 or 1 we cannot make such an observation because we wont know the velocity or position. If we used disjunction this relationship would not be observed since one proposition having a probability of 0 would not cause the resulting sentence to also have a probability of 0. In conjunction it is the case since 0 multiplied by any other number is always 0.

Definition 4.0.7. Inclusive disjunction is another way of saying or, as exclusive disjunction. However, by adding AB it adds the parts of A+B which normally would have been equal to 0, hence it does not mod out identical truth values.

[23]All connectives are formed from addition and multiplication. Addition is symmetry preserving. In multiplication, 0 causes many values to map to 0, so it's symmetries are not invertible.

$$\text{A}\vee\text{B}=\begin{bmatrix} y & 1 & 1 & 1 \\ y & 1 & 1 & 1 \\ y & 1 & 1 & 1 \\ 0 & x & x & x \end{bmatrix}=\text{A}+\text{B}+\text{AB}.$$

Example 4.0.2. Consider the sentence: Either you're 'observing' this paper, or I am.
Let A=You're observing this paper, B=I am observing this paper.
There is no doubt that A ∨ B is false only when neither of us are observing the paper. However, by observing, it must be meant that there are 4 states in which this observation can manifest in a truth space.
Observing is 1(true) if someone had directly read it, and of course is thinking about it. Observing is x(neither) if it has never been read, but it has been conceived of in an indirect way. Observing is y(both) if the paper has been read but the content is not being thought about further. Finally, observing is 0(false) if the paper has never been read, and has not even been conceived.
For each of us, we can observe the paper up to those truth values. Saying you observed or I did(or us both) is to say $\text{A}\vee\text{B}=\begin{bmatrix} y & 1 & 1 & 1 \\ y & 1 & 1 & 1 \\ y & 1 & \underline{1} & 1 \\ 0 & x & x & x \end{bmatrix}$. Hence as long as I have not forgotten about the paper, it is always observed to be not false. A nice way of saying this is the paper is existent since it is never false. It could be x, y, or 1. In fact, of course while I'm writing now it's true, hence the sentence is true. If I stop writing, and go for a walk, my perspective goes to x. Of course, as you're reading this and I am off on my walk, you could be reading it but not thinking of it at all, so together our combined(inclusively disjunctive) observation makes the sentence true. I've underlined this case in the matrix above. Just as in the previous example, we could even consider a probability distribution on the propositions which considers the probability of each truth state arising. What is the chance that at some time, t, I am reading or writing and apprehending what I am reading or writing? This is the probability for 1. However, does 0 ever show up from my perspective; is the proposition I am observing this paper ever false from my perspective? Certainly it isn't arising now as I write, but I cannot be so certain that I will also be apprehending as you're observing this. It may even be fair to suggest that often I am neither apprehending nor am I reading or writing. If it never did, what would this tell us? To consider all of these scenarios it is helpful to consider probability matrices for the states of observation. There are two types of probability matrices. Those which distribute the probability of 1(100%) along all states, and a matrix which distributes two states of probability along each absolute or relative diagonal. Go to quantification for more on this.

Definition 4.0.8. Implication has the form: if A then B. In Boolean logic, we know this is equal to not A or B, using inclusive disjunction. A then B is the value of B when the values are conjugate pairs. Doing this gives the same equivalence as Boolean logic, and it allows us to easily construct the bi-conditional.

$$\text{A}\to\text{B}=\begin{bmatrix} 1 & 1 & y & 1 \\ x & x & 0 & x \\ 1 & 1 & y & 1 \\ 1 & 1 & y & 1 \end{bmatrix}=A_1 \vee B$$

Definition 4.0.9. Biconditional If A and B have the same truth value, then A if and only if B is true. It is false if A and B are conjugates, A if and only if B is equal to 'If A then B and If B then A'.

$$\text{A} \iff \text{B}=\begin{bmatrix} 1 & y & y & 1 \\ x & 0 & 0 & x \\ x & 0 & 0 & x \\ 1 & y & y & 1 \end{bmatrix}=(A \to B)(B \to A) = (A+B)_1.$$

Proof. $\begin{bmatrix}1&1&y&1\\x&x&0&x\\1&1&y&1\\1&1&y&1\end{bmatrix}\begin{bmatrix}1&y&1&1\\1&y&1&1\\x&0&x&x\\1&y&1&1\end{bmatrix}=\begin{bmatrix}1&y&y&1\\x&0&0&x\\x&0&0&x\\1&y&y&1\end{bmatrix}=(A+B)+1=(A+B)_1$ □

5 Equivalences

Prop 5.0.4. ***Commutativity of*** V_4

A+B=B+A. This is easy to prove using our notation. A+B= $\begin{bmatrix}0&x&x&0\\y&1&1&y\\y&1&1&y\\0&x&x&0\end{bmatrix}$. *Taking the lower left hand corners of each B block, we construct the B block for B+A. Notice that the main corners remain the same? The (0,A) corner for B+A is* e_0. *Constructing all corners of B we get B+A=* $\begin{bmatrix}0&x&x&0\\y&1&1&y\\y&1&1&y\\0&x&x&0\end{bmatrix}$.
We can do the same thing for $A\odot B$. $A\odot B$*=*$B\odot A$*, hence the same can be done for* $A\vee B$*, but not for* $A\rightarrow B$*. That is* $A\rightarrow B\neq B\rightarrow A$.

Prop 5.0.5. *It is easy to show* $A_1\vee B=A\rightarrow B$*, just as we would have in propositional logic.*

Proof. $A_1\vee B=A\rightarrow B\Rightarrow\begin{bmatrix}x&x&0&0\\x&x&0&0\\1&1&y&y\\1&1&y&y\end{bmatrix}\vee\begin{bmatrix}y&1&y&1\\0&x&0&x\\y&1&y&1\\0&x&0&x\end{bmatrix}=\begin{bmatrix}1&1&y&1\\x&x&0&x\\1&1&y&1\\1&1&y&1\end{bmatrix}$ □

Prop 5.0.6. ***Law of Transposition****:* $B\rightarrow A=A_1\rightarrow B_1$.

Proof. Take the proposition $B\rightarrow A$ and rearrange the entries so that it is read with A on the outside and B on the inside. $\begin{bmatrix}1&y&1&1\\1&y&1&1\\x&0&x&x\\1&y&1&1\end{bmatrix}$.
$A_1\rightarrow B_1$ means to take the symmetric rotation, 1 negation on both propositions. When performing operations on a group of premises, one must make sure the order of each block is built in the same way. Hence, again, $A\rightarrow B\neq B\rightarrow A$. □

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Prop 5.0.7. ***Rules for Natural Deduction***
1) ***Conjunction Exploitation***
$AB\therefore A(orB)$
Given A and B, then we can deduce either A or B.
2) ***disjunction Exploitation*** $A_n\vee B_m$
$A_{\bar{n}}$*(or* $B_{\bar{m}}$*)*
$\therefore B_m$*(or* A_n*)*

3) ***Conditional Exploitation***
$A\rightarrow B,A\therefore B$*. Given* $A\rightarrow B$*, and the predicate A, then we may conclude B.*

Proof. See $A\rightarrow B$, if A=1, the upper right corner, and we have e_0=B. □

*4) **Biconiditional Introduction***
$A \to B, B \to A \therefore A \iff B$
This has been proved in definition 4.0.9.
*5) **Disjunction Exploitation***
$A \vee B, A \to C, B \to C \therefore C$

Proof. $1.A \vee B = A + B + AB$
$2.A \to C = A_1 \vee C = A_1 + C + A_1C$
$3.B \to C = B_1 \vee C = B_1 + C + B_1C$
4. show C
5. $[2][3] \Rightarrow A_1B_1 + A_1C + A_1B_1C + B_1C + C + B_1C + A_1B_1C + A_1C + A_1B_1C$
6. $= A_1B_1 + A_1B_1C + C$
7. $= (A_1B_1) \vee C$
8. $= (A \vee B)_1 \vee C$
9. $\therefore C$

□

Prop 5.0.8. *Extended **De'Morgans Laws***
Let A, B both be represented by some symmetry of e_0, then
1)$(A + B)_n = A + B_n = A_n + B$
2)$(AB)_n = A_1 \vee B_1 + \overline{n}$
3)$(A \vee B)_n = A_1B_1 + \overline{n}$
The following proof illustrates algebraically (2), though one can clearly see that it simultaneously would prove (3), and 1 requires no proof other than inspection.

Proof. $(AB)_n = A_1 \vee B_1 + \overline{n}$[24]$=A_1 + B_1 + A_1B_1 + AB + AB_{\bar{n}} + A_{\bar{n}}B + A_{\bar{n}}B_{\bar{n}}$[25]$=A + B + (A+1)(B+1) + AB + A(B + \bar{n}) + (A + \bar{n})B + (A + \bar{n})(B + \bar{n})=$
$=A + B + AB + A + B + 1 + AB + AB + A\overline{n} + AB + B\overline{n} + AB + A\overline{n} + B\overline{n} + \overline{n}=$
$=1 + AB + \overline{n} = AB + n = (AB)_n$ □

Prop 5.0.9. *More Rules For Natural Deduction*
*(1)**Conditional disjunction*** $A \to B = A_1 \vee B$
Look at $A_1 \vee B$ and compare this to $A \to B$. $A_1 \vee B$ is the symmetry taking the larger block structure and reflecting them along their conjugate element: 1 goes to 0, 0 goes to 1, x goes to y, and y goes to x. The result is $A \to B$.
*(2)**Conditional Negation*** $(A \to B)_n = (A_1 \vee B)_n = AB_1 + \overline{n}$

Proof. By proposition 2.2.6(1), above, $(A \to B)_n =$
$= (A_1 \vee B) = (A_1 + B + A_1B)_n$
$= (A_1 + B) + (A_1B)_n =$
$= (A_1 + B) + (A \vee B_1 + \overline{n}) =$
$= (A_1 + B) + (A + B_1 + AB_1 + \overline{n}) = AB_1 + \overline{n}$ □

*(3)**Bi-conditional negation*** $(A \iff B)_n = (A + B)_{\overline{n}}$
Proof by inspection.
*(4)**Conditional Exploitation **** $A \to B, and B_n \Rightarrow A_1B_n + ABB_n$

[24] $\overline{n} = (A + A_{\overline{n}})(B + B_{\overline{n}})$
[25] $AB + n = AB_n + BA_n + A_nB_n$

Proof. $(A \to B)B_n =$
$=(A_1 \vee B)B_n = (A_1 + B + A_1B)B_n =$
$=(A_1B_n + BB_n + A_1BB_n) = (A_1B_n) + BB_n(1 + A_1) =$
$=A_1B_n + ABB_n$ □

Hence, if n=1 $\Rightarrow A_1B_1$
*(5)**Bi-conditional Exploitation*** $A \iff B, and A_n(or B_n) \Rightarrow AA_n + A_nB_1$

Proof. $(A \iff B)(A_n) =$
$= (A + B)_1A_n =$
$= (A + B_1)A_n =$
$= AA_n + A_nB_1.$
Had we let $(A + B)_1 = (A_1 + B)$ instead, then
$(A + B)_1A_n = (A_1 + B)A_n = A_1A_n + BA_n$

□

If n=1 $\Rightarrow A_1B_1$
*(6)**Disjunction Exploitation*** $A \vee B$*, and* $A_n(or B_n) \Rightarrow A_n(AB_1 + B)$

Proof. $(A \vee B)A_n =$
$= (A + B + AB)A_n = AA_n + BA_n + AA_nB =$
$= AA_nB_1 + BA_n =$
$= A_n(AB_1 + B)$ □

If n=1 $\Rightarrow A_1B$

5.1 Deduction Examples

The following arguments are taken from Deduction, by Daniel Bonevac. For all negations, since we don't know the negation, we will treat it generally, hence the following examples really are illustrating the different behavior of negations through a proof structure.

In propositional logic, using a Boolean truth space, we create a contradiction by having a proposition p and p_1(not p). The conjunction of p and not p is false, which tells us that the multiple of a proposition and a symmetry of it will let us infer the validity of the system.

Example 5.1.1. Given not q or not p, using inclusive disjunction. Show if p then not q.

Proof. 1) $p_n \vee q_n = p_{\overline{n}} \to q_n$ from prop 2.2.6(1). If n=1, as we have for Boolean logic, then $\overline{n}$=0, hence we have $p \to q_1$, as we would with Boolean logic. □

Example 5.1.2. 1) $q \iff (qp)$
2) show $q \to p$
3) q [Assume by conditional proof]
4) $q \to (qp)$ [from 1 and bi-conditional exploitation]
5) qp [from 4 and 3]
6) p [from 5, conjunction exploitation]
Here we have the same result as we would from Boolean logic, of course this is because there were no negations to shift our perspective from a dual system.

Example 5.1.3. 1)$p \iff (r \vee s)$
2)p_n
3)show r_n

Proof. multiplying 1 and 2 $\Rightarrow$
4) $p_1(r \vee s)_n + (r \vee s)(r \vee s)_n$
5)factor: $(r \vee s)_n(p_1 + (r \vee s))$
6)$(r \vee s)_n = r_n s_n + \overline{n}$
7)$r_n(s_n + \overline{n}/r_n$
8)r_n □

Example 5.1.4. 1)$p \vee (r \vee q))$
2)r_n
3) show $p \vee q$

Proof. 4)$(p \vee q)_n$ [Assumption by indirect proof]
5)$(p_1 q_1 + \overline{n})$.[Disjunctive negation exploitation]

The next line requires factoring a p_1, look at the end of section 3 at the division algorithm for an idea how this may be done.

6)$(p_1 q_1 + \overline{n}) = p_1(q_1 + \overline{n}/p_1)$
If the argument were valid, then $p_1 \neq 0$, and we would not divide by 0.
7)p_1 [conjunction exploitation]
8)$(r \vee q)$ [disjunction exploitation, using 1 negation]
9)$r_n(rq_1 + q)$ [from line 2 and 8]
10)$rq_1 + q$
11)$q_1(p_1 + \overline{n}/q_1)$ [In the same way we have line 6]
12)q_1[conjunction exploitation line 11]
13)rq_1[disjunction exploitation line 12 and line 10]
14)r
15)$rr_n = r^{\bar{n}}$

Recall we started with an indirect proof, we have shown a partial contradiction, $r^{\bar{n}}$.Therefore, we know $(p \vee q) + (p \vee q)_n r^{\bar{n}}$.

16)$p_1 r^{\bar{n}} = rp_1^{\bar{n}}$.
17)$\bar{n} + p^{\bar{n}}$ This is our contradictory element.
18)[AIP][$\bar{n} + p^{\bar{n}}$]+[Show]
19)$(p \vee q)_n(\bar{n} + p^{\bar{n}})$=
20)$(p_1 q_1 + \overline{n})(\bar{n} + p^{\bar{n}})$=
21)$\bar{n}p_1 q_1 + \bar{n} + p^{\bar{n}} p_1 q_1 + \bar{n}p^{\bar{n}}$=
22)$\bar{n}(p+1)q_1 + \bar{n} + p^{\bar{n}} p_1 q_1 + \bar{n}p^{\bar{n}}$=
23)$p^{\bar{n}} q_1 + q_1 \bar{n} + \bar{n} + p^{\bar{n}} p_1 q_1 + p^{\bar{n}}$=
24)$p^{\bar{n}} q_1 + q^{\bar{n}} + p^{\bar{n}} q_1 p_1 + p^{\bar{n}}$=
25)$p^{\bar{n}} q_1 p + q^{\bar{n}} + p^{\bar{n}}$=
26)$p^{\bar{n}} q_1 + q^{\bar{n}} + p^{\bar{n}}$=$\bar{n}p(q+1) + q\bar{n} + p\bar{n} = \bar{n}pq + \bar{n}p + \bar{n}q + \bar{n}p =$
27)$p^{\bar{n}} 1 + q^{\bar{n}}$=$\bar{n}q(p+1) = \bar{n}qp_1$
$[AIP][\bar{n}qp_1] + [show]$
28)$\bar{n}qp_1 + p + q + pq$=
29)$\bar{n}(p+1)q + p + q + pq$=
30)$\bar{n}pq + \bar{n}q + p + q + pq$=
31)$pq(\bar{n}+1) + (\bar{n}+1)q + p$
32)$p \vee (\bar{n}+1)q$

$33)p \vee nq$

□

Example 5.1.5. $1)p \rightarrow r = p_1 \vee r = p_1 + r + p_1 r$
$2)(rp)_n = r_1 \vee p_1 + \overline{n} = r_1 + p_1 + r_1 p_1 + \bar{n}$
3)show p_n
$4)p_{\bar{n}}$[Assumption by indirect proof]
5)[1 and 2] $\Rightarrow r_1p_1 + r_1r + r_1p_1r + p_1p_1 + p_1r + p_1p_1r + r_1p_1p_1 + r_1p_1r + r_1p_1p_1r + \overline{n}(p_1 \vee r)$
$6)(r_1p_1 + p_1 + p_1r_1 + \overline{n}(p_1 \vee r) =$
$7)p_1 + \bar{n}(p_1 \vee r) =$
$8)p_1 + \bar{n}(p_1 + r + p_1 r) =$
$9)p_1 + \bar{n}p_1 + \bar{n}r + \bar{n}p_1 r =$
$10)np_1 + \bar{n}rp =$
$11)n(p+1) + (n+1)rp =$
$12)np + n + npr + rp =$
$13)npr_1 + (rp + n).$
Multiply by $p_{\bar{n}}$ to get a contradiction from line 13.
$14)(npr_1 + (rp+n))(p_1 + n) =$
$15)p_1(npr_1) + p_1(rp) + np_1 + npr_1 + npr + n =$
$16)n(p+1) + np(r+1) + npr + n =$
$17)n(p+1) + nor + np + npr + n =$
$18)np + n + npr + np + npr + n =$
19)0. *It's the holy grail!*
20)∴[show]+0[AIP]=[show]=p_n

Example 5.1.6. $1)p \iff r$
$2)(u \rightarrow r)_n = ur_1 + \bar{n}$
3)show p_n
$4)p[AIP]$
$5)r$
$6)r_1$[factored r_1 from 2]
$7)rr_1 = 0$
8)∴ p_n

Example 5.1.7. $1)p \iff r$
2)show $(p_n \vee q_n)_n \rightarrow r$

Proof. 3)[assumption by conditional proof] $(p_n \vee q_n)_n$=
4)=$p_{\overline{n}}q_{\overline{n}} + \overline{n}$=
5)=$p_{\overline{n}}(q_{\overline{n}} + \overline{n}/p_{\overline{n}})$
6)$p_{\overline{n}}$
[6][1]
$7)pp_{\bar{n}} + p_{\bar{n}}r_1 =$
$8)p_{\bar{n}}(p + r + 1) =$
$9)p_{\bar{n}}p + p_{\bar{n}}r + p_{\bar{n}} =$
$10)p(p_1 + n) + (p + \bar{n})r + p + \bar{n} =$
$11)np + pr + \bar{n}r + p + \bar{n} =$
$12)p(r + n + 1) + \bar{n}r_1 =$
$13)pr + \bar{n}p + \bar{n} + r =$
$14)rp_1 + \bar{n}p_1 =$

$15) p_1 r_{\bar{n}} =$
$16) r_{\bar{n}}$ This example is interesting because we don't show r from the conditional, but only have managed to show $r_{\bar{n}}$. If n was the Boolean negation, 1, then $\bar{n}$ is 0 and we have shown r.

□

6 Higher order maps

Definition 6.0.1. A map from a truth value to an ineffable truth state is an **expansion**

This map has been used to extend $\mathbb{Z}_2$ to V_4. Furthermore, we extend V_4 using this map to U_4, the set of ineffables and tautologies.

Definition 6.0.2. A map from a a truth state represented by an element of U_4 to a truth value in V_4 is a **contraction.** A map from e_0 to $\underline{v}, \underline{v} \in \underline{D_4}$, is a contraction onto v from e_0.

Prop 6.0.1. *A specific classification of contraction and expansion maps are* $0 - e_0, x - e_x, y - e_y, 1 - e_1, \rho - e_\rho, \overline{\rho} - e_{\overline{\rho}}, \tau - e_\tau, \sigma - e_\sigma$. *Where v-*$e_v$ *means a map which could go either way, as an expansion or contraction. We will call the expansion map the integral, and the contraction map the standard derivative.*

The above relation -, which is the class of contraction and expansion maps above, is a recursion. The dimension of the matrices is a ratio of the number of copies in each recursion. This allows us to view the relation as a fractal.

Example 6.0.8. Using the above expansion map, with our initial condition $v_0 = e_0$, then v_1 is defined by 0 to e_0, y to e_y, x to e_x, 1 to e_1. $v_1 = \begin{bmatrix} 0 & x & x & 0 \\ y & 1 & 1 & y \\ y & 1 & 1 & y \\ 0 & x & x & 0 \end{bmatrix}$. This looks like the truth space for a sentence 'A or B', using exclusive disjunction. Let $v_1 = u_0 + u_1$. Inductively it's easy to show that we will continue adding proposition for each expansion. So, $v_2 = u_0 + u_1 + u_2$.
There are 4 truth states so there are 4 copies.

Example 6.0.9. Define a map $M : e_0 \mapsto \begin{bmatrix} \underline{y} & \underline{1} \\ e_0 & \underline{x} \end{bmatrix}$
Let $F_k : M^k(e_0)$. $\lim_{k \to +\infty}$ creates a space with only one 0, at the 0 state.
This map, like the map above creates a space which could be rewritten in terms of 'virtual' objects(these were the u's). However, such a closed form is not so easily constructed.
If $F_0 = e_0$, then $F_1 = \begin{bmatrix} \underline{y} & \underline{1} \\ e_0 & \underline{x} \end{bmatrix}$, and $F_2 = \begin{bmatrix} \underline{y} & \underline{y} & \underline{1} & \underline{1} \\ \underline{y} & \underline{y} & \underline{1} & \underline{1} \\ \underline{y} & \underline{1} & \underline{x} & \underline{x} \\ \underline{0} & \underline{x} & \underline{x} & \underline{x} \end{bmatrix}$. We can construct a virtual geometric interpretation for F_∞ which looks like a triangle constructed from the square of V_4, by drawing a diagonal line between x and y. Considering two elements, whose structure is like this, which are summed, we inscribe the second element inside the first like we've done with V_4. However, there are 4 triangles created by doing this, only the 3 which each share one of the original vertices are of importance here. Furthermore, such a construction can be done by identifying the mid point between two nodes as the sum of those nodes. Hence, the value at one of the vertices is always 0. We will denote this triangle as E_0 which can be carefully represented by the matrix $\begin{bmatrix} \underline{y} & \underline{1} \\ e_0 & \underline{x} \end{bmatrix}$.

Example 6.0.10. If $\int e_0 = u_1 + u_2$ and $\int \underline{0} = \begin{bmatrix} e_0 & e_0 \\ e_0 & e_0 \end{bmatrix}$.
Then there exists a symmetry F such that $F(\int \underline{0}) = u_1 + u_2$.

$\Delta F(\int \underline{0}) = u_0$.
$\Delta\Delta F(\int \underline{0}) = 0$.
F is defined by identifying block structures in some 4x4 matrix. The center block structure, which has all conjugate elements in it is mapped to the truth state 1. The four corners are mapped to the 0 state. The 2 paths from 0 to x and y to 1 are grouped and mapped to the x state. The two paths from 0 to y and x to 1 are grouped and mapped as one block to the y state.
From the above example we see an elegant relationship between $\underline{0}$ and e_0. The reason for studying this example is simple: the derivative of $\underline{0}$ is not defined, only the derivative from a state matrix to a negation matrix. The map F allows us to transition between the two, and thus show that the derivative of each is both equal to 0.

Prop 6.0.2. *Integration is a expansion map, with the same index recursion we have used for division, and the reoccurring fractal patterns. Differentiation is a contraction map.*

The map - which defined the standard contraction and expansion maps is identical to integration and differentiation of the truth space.

6.1 Ineffable Partial Negations

Given e_0 (or some symmetry) a partial reflection of e_0 is constructed by a tautology expansion map, mapping each element of the truth space to a tautology of that element, and then translating the elements in the direction of the fold. Translating from 0 to the base of the negation. For all partial negations, the flow of truth elements is, for convention, from relative to absolute diagonals. Reflections are written $e_{r/2^k}, k \in [0, 1, 2, ..., K)$

Example 6.1.1. $e_{x/2} = \begin{bmatrix} y & y & 1 & 1 \\ y & y & 1 & 1 \\ 0 & 0 & x & x \\ 0 & 0 & x & x \end{bmatrix}_{x/2} = \begin{bmatrix} y & 1 & 1 & y \\ 1 & y & y & 1 \\ 0 & x & x & 0 \\ x & 0 & 0 & x \end{bmatrix}$

Rotations of a space, e_0, are written as $e_{\pi/2^n}$, n $\in [0, 1, 2, ..., N)$

Rotating counter clockwise, the previous truth state's relative values are mapped to the absolute values in the next state.

It is rotated such that for even n, the entire relative or absolute diagonal is mapped. While for odd n, each truth value is mapped one at a time, such that in 2 iterations it would map the entire diagonal like the rotation which is composed of an even n.
The dimension of the ineffable parts is equal to $n/2 + 1$, rounded down.

Example 6.1.2. $e_{\pi/4} = e_{\rho/2} = \begin{bmatrix} y & 1 & 1 & x \\ 1 & y & x & 1 \\ 0 & y & x & 0 \\ y & 0 & 0 & x \end{bmatrix}$

Example 6.1.3. consider $e_\rho + e_{\rho/2} = \begin{bmatrix} 1 & 1 & x & x \\ 1 & 1 & x & x \\ y & y & 0 & 0 \\ y & y & 0 & 0 \end{bmatrix} + \begin{bmatrix} y & 1 & 1 & x \\ 1 & y & x & 1 \\ 0 & y & x & 0 \\ y & 0 & 0 & x \end{bmatrix} = \begin{bmatrix} x & 0 & y & 0 \\ 0 & x & 0 & y \\ y & 0 & x & 0 \\ 0 & y & 0 & x \end{bmatrix} = \underline{\rho'/2}$ This is the ineffable symmetry defined going the opposite direction, that is from one states relative to its own absolute, from that absolute to the next state's relative position.

Example 6.1.4. $e_{\rho/2} + e'_{\rho/2} = \begin{bmatrix} \underline{x} & y \\ \underline{y} & \underline{x} \end{bmatrix} = \underline{\rho} = \pi/2$ the angle these two elements are separated by.

Example 6.1.5. $e_{\rho/8} = \begin{bmatrix} \underline{y} & \begin{bmatrix} y & 1 \\ 1 & y \end{bmatrix} & \underline{1} & \underline{1} \\ \underline{y} & \underline{y} & \begin{bmatrix} 1 & x \\ x & 1 \end{bmatrix} & \underline{1} \\ \underline{0} & y\underline{\rho} & \underline{x} & \underline{x} \\ \underline{0} & \underline{0} & x\underline{\rho} & \underline{x} \end{bmatrix}$

Example 6.1.6. $e_{\overline{\pi/4}} = e_{\overline{\rho/2}} = \begin{bmatrix} y & 0 & 1 & y \\ 0 & y & y & 1 \\ 0 & x & x & 1 \\ x & 0 & 1 & x \end{bmatrix}$

Example 6.1.7. $e_{\rho/4} = \begin{bmatrix} y & 1 & 1 & 1 \\ y & y & x & 1 \\ 0 & y & x & x \\ 0 & 0 & 0 & x \end{bmatrix}$

Prop 6.1.1. *A partial negation n/2 is defined by $v_1 + v_2 + e_0\tau + e_o\sigma$ for some virtual propositions v_1 and v_2.*
Going in the opposite direction, $v_1 + v_2 + e_0\sigma + e_0\tau$.

In fact all partial negations can carefully be defined as above. A negation of n/8 is defined by $v_1 + v_2 + v_3 + e_0\tau + e_0\sigma$.

6.2 division

Theorem 6.2.1. *Using an expansion map,*
Given A,B represented by e_0, then $B/A = \begin{bmatrix} B/y & B_1 \\ B/0 & B/x \end{bmatrix}$
$B/0 = \begin{bmatrix} e_y & e_1 \\ e_0 & e_x \end{bmatrix}$
$B/1 = \begin{bmatrix} \underline{y} & \underline{1} \\ \underline{0} & \underline{x} \end{bmatrix}$

In the section on arguments, we used this function to let us factor truth states from symmetry objects.The other 8 elements are the elements of U_4 in some orientation. These 4 matrices can be arranged in 4 space to form the vertices of a hyper cube.

7 Quantification

Prop 7.0.1. *A sequence can be defined by an expansion map. Furthermore, such a map and the sequence can define a complex number z such that we assign to each $v \in V_4$ a complex number. 0=0, x=1, y=i, and 1=1+i. Then, for each expansion we divide by 2^k where k starts at 0 and goes to infinity.*

Example 7.0.1. let $\{v_i\} = \{1, 1, 1, ..., 1\} \Rightarrow S = \sum_{i=0}^{k} v_i/2^i = 1 + 1/2 + 1/4 + ... + 1/2^k \Rightarrow \lim_{k\to+\infty} S = 2$

Example 7.0.2. let $\{v_i\} = (x)_{\tau^k} = \{x, y, x, ...,\} = \{1, i, 1, i, ...\} \Rightarrow S = \sum_{i=0}^{k} v_i/2^i = 1 + i/2 + 1/4 + ... \Rightarrow$ $\lim_{k\to+\infty} S = 4/3(1 + i/2)$

Definition 7.0.1. The determinant of e_0(or any symmetry) is written $\det(e_0)$=(x+y)(0+1)=1.

Definition 7.0.2. The trace of e_0(or any symmetry) is written tr(e_0)=(1+0)+(x+y)=1+1=0

Prop 7.0.2. *For some* $v \in \{|V_4|\}$, *tr(v)+det(v)=1.*
For some $v \in \{\underline{V_4}\}$, *tr(v)+det(v)=0.*

Definition 7.0.3. A coefficient matrix C assigns to each truth state of a proposition a complex number z. We write these complex numbers as a,b,c, and d
let C= $\begin{bmatrix} d & c \\ a & b \end{bmatrix}$

Prop 7.0.3. *If a proposition P=e_0, then CP means the coefficient of 0 is a, of x is b, of 1 is c, of y is d. Hence if a is 0 then P is never false.*

Prop 7.0.4. *If q=*$\begin{bmatrix} e_y & e_1 \\ e_0 & e_x \end{bmatrix}$ *then the state of q is determined by a, b, c, d .*

Here, C is a coefficient matrix on states not necessarily values.

Example 7.0.3. letting C be all 0 except the 1 state would tell us q always has a true state, but that in itself represents e_1 without discrimination of values. If values were to be apprehended, then for all states except 1 all values are 0, while at 1 the distribution is through the entire set.
$Cq = \begin{bmatrix} 0e_y & 1e_1 \\ 0e_0 & 0e_x \end{bmatrix}$

Prop 7.0.5. *C can be a probability matrix such that det(C)=1. Hence* $C \in \{|V_4|\}$ *is a probability matrix with det=1.*

Prop 7.0.6. *C can be a probability matrix such that tr(C)=1*

These two probability matrices allow us to discuss rather interesting properties of systems.

Example 7.0.4. Let a single particle be described by 2 propositions, velocity and position propositions. Consider a particle is moving at s meters/second. Let this be A.
A particle is observed at position z. Let this be B.
Both A and B have probability matrices C_A and C_B such that $\underline{1} = C_A + C_B$. C_A is the probability of observing the particle move at that velocity, and C_B is the probability of observing it at position z. Are these probability matrices of the type tr(C)=1, or det(C)=1? Given that it is det(C)=1, then:

Let $C_A = \begin{bmatrix} 2/3 & 1 \\ 0 & 1/3 \end{bmatrix}$
Consider the sentence AB, given our quantified system, we consider $C_A A C_B B$=$C_A C_B AB$=
$\begin{bmatrix} 2/9 & 0 & 1/3 & 0 \\ 2/3 & 4/9 & 1 & 2/3 \\ 0 & 0 & 1/9 & 0 \\ 0 & 0 & 1/3 & 2/9 \end{bmatrix}$BA

In general, there are up to four quantum numbers for a particle, so a particle should be capable of being determined with no more than 4 distinct matrices and connectives, and quantified coefficient matrices.

Prop 7.0.7. *A matrix D such that D describes the behavior of a system, such as space time. Combined with a truth space, or series of connectives allows us to discretely manipulate the system through the truth space. D is not restricted necessarily by the trace or determinant functions. In fact, a value in D can be any complex number.*

8 Appendix

8.1 Philosophical examples

Example 8.1.1. If ∃ P, Atman, such that ∀ v ∈ E[26] P and v have the essential nature of P. We can write this as
$Pv \to P = (Pv)_1 \vee P = (Pv)_1 + P + P(Pv)_1 = P_1 \vee v_1 + P + (P_1 \vee v_1)P = P_1 + v_1 + P_1v_1 + P + (P_1 + v_1 + P_1v_1)P = P_1 + v_1 + P_1v_1 + P + Pv_1 = 1 + v_1 + v_1(P_1 + P) = 1 + v_1 + v_1 = 1$. Therefore this is a valid argument. Though, it does clearly state if there is such a P. So now we can investigate based on this valid statement, as we did in section 2 with respect to Atman. In doing so though, we multiply P(atman) and v, here we are looking for what unifies them. What do P and v have in common except 0? The product of an infinite series of v and this P will eventually give just the essential nature of P, which turns out to be 0. Of course, 0 → P is always valid, regardless of the state of P. However, while analyzing the condition Pv we see that P must be 0, even if all v were never 0. The only way it could not be the case is if all v were in fact 1, and P was never 0. This would suggest that there is equivalently only 1 state, for which P and v share, but it would also suggest that all 'existing' things are absolutely existing, while atman is not necessarily absolutely existing. However, we know that all phenomena can be described as an interaction between x and y. If two phenomena have as their base Atman, then the conjunction of these two with Atman ought to leave the essential nature of Atman. But, the conjunction of x and y is always 0, thus, if Atman also exists equally or greater than the phenomena, then the essential nature of Atman must have truth value 0.
We haven't determined if any of these phenomena, including Atman, actually have truth values. Thus, it could be that Atman is more appropriately described by e_0.

Example 8.1.2. Recall, one of our motivating examples was the discussion between Vaccha and the Buddha on the nature of the Arhat.
First, we use the proposition P='The arhat exists after death'. Letting P be represented by e_0, we consider $P + P_1 + P_x + P_y = \underline{0}$
Vaccha expects, as he asks each symmetry of P, to get a true statement for one arrangement. He does not expect that each propositional arrangement will be simultaneously rejected by the Buddha. If none were rejected the sentence itself would not be valid, so Vaccha does expect some of these arrangements to be false. The sentence $P + P_1 + P_x + P_y$ is not valid if all propositions are true. For this reason as Vaccha states each proposition, he says that all other arrangements are false.
By rejecting the sentence Vaccha could consider that the problem lies in that phrase that all other views are false.
Even if he considered other views which might not be false, the Buddha would fundamentally reject the concept since Vaccha holds onto the constructed views of self and existence.
The first problem to analyze is in how he asked the question, as an exclusive disjunctive sentence: If he rearranged the propositions to be the answer the Buddha gave, i.e "It is not the case the Arhat exists after death"=P, and considered a similar exclusive disjunction of these 4 symmetries, the sentence would still not be valid. If Vaccha assumed that each new proposition was true[27] it would still be of the form $P + P_1 + P_x + P_y$ which is not valid. To be valid, some of these arrangements would still have to be rejected. This would suggest that the negation used by the Buddha is conditional to the question Vaccha is asking, and it could then be that the sentence ends up being P+P+P+P, if P represented "It is not the case that the arhat exists after death". This doesn't tell us much about the nature of the Arhat. But it does suggest that the nature of the Arhat is conditional to how we conceive of it. Hence the inferred sentence is valid if and only if we consider different negations in response to Vaccha's questions.

[26] Pertaining to all existing things: There is not one thing which can be conceived of outside of this E. I am flying does not mean in all realms of conception I am flying, but that the one for which the imagined conception that I am flying am I flying. For any v ∈ E, v may not reciprocally refer back to E. Clearly we may conceive of virtual place holders in E which can negate E.

[27] By the Buddha's rejection of the previous statement. Notice this is also a dualistic view

For now, let's conclude that the sentence itself is not valid, and that the Buddha has rejected all 4 statements of Vaccha.

It could be that the principal problem was the use of exclusive disjunction, or that all other views were false. Hence, here we would use inclusive disjunction instead. However, this arrangement is still an attempt at reevaluating the above interpretation given the reassignment of the Buddha's answer to Vaccha as our premises. So, we instead consider the sentence $(P+P_1)\vee(P_x+P_y)$, which is valid, and can accommodate the reassignment of the Buddha's answer to the proposition P. The result ends up being essentially meaningless because it still supposes that there is a self, and that there is existence pertaining to self of some form.

When the Buddha says it is not the case, he is simply absolutely rejecting the statement that has arose: it is not a specific negation, it does not imply a symmetry *is* absolutely the case,though that is what we had abstractly considered above. We showed that despite this possibility, it would not yield meaningful results. Another way of perceiving the rejection is that it is ineffable, the true nature of the Arhat is not contained within the truth space. Much like our final conclusion for 8.1.1, which was that Atman had truth value of 0, we cannot be sure that it has any ultimate existence so we conceive of it as e_0. In the same way the nature of self and existence can be characterized by the ineffable of e_0.

If we instead assigned P='The arhat exists after death' the ineffable e_0, then we consider the same sentence $P+P_1+P_x+P_y$. Since each is really an ineffable, treating each of these as distinct propositions we get an equivalent expression: $v^1+v^2+v^3+v^4$. The negations cancel out nicely, and we're left with the sum of 4 propositions. However, this is equivalent in structure to e_0 given the recursive map in proposition 6.0.1. Hence, the answer to the question becomes similar to the nature of the self. In the previous example we considered Atman, and suggested the nature of Atman was 0. In reality, we notice that Atman is ineffable, and not ultimately false.

8.2 Conjunction Negation Table

n	e_0	e_x	e_y	e_1
0	0	0	0	0
x	$\begin{bmatrix}0 & x\\0 & x\end{bmatrix}$	$\begin{bmatrix}x & 0\\x & 0\end{bmatrix}$	$\begin{bmatrix}0 & x\\0 & x\end{bmatrix}$	$\begin{bmatrix}x & 0\\x & 0\end{bmatrix}$
y	$\begin{bmatrix}y & y\\0 & 0\end{bmatrix}$	$\begin{bmatrix}y & y\\0 & 0\end{bmatrix}$	$\begin{bmatrix}0 & 0\\y & y\end{bmatrix}$	$\begin{bmatrix}0 & 0\\y & y\end{bmatrix}$
1	e_0	e_x	e_y	e_1

n	e_0	e_x	e_y	e_1
ρ	$\begin{bmatrix}0 & y\\0 & x\end{bmatrix}$	$\begin{bmatrix}x & y\\0 & 0\end{bmatrix}$	$\begin{bmatrix}0 & 0\\y & x\end{bmatrix}$	$\begin{bmatrix}x & 0\\y & 0\end{bmatrix}$
$\bar{\rho}$	$\begin{bmatrix}y & x\\0 & 0\end{bmatrix}$	$\begin{bmatrix}y & 0\\x & 0\end{bmatrix}$	$\begin{bmatrix}0 & x\\0 & y\end{bmatrix}$	$\begin{bmatrix}0 & 0\\x & y\end{bmatrix}$
τ	$\begin{bmatrix}y & 0\\0 & x\end{bmatrix}$	$\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}$	$\begin{bmatrix}0 & 0\\0 & 1\end{bmatrix}$	$\begin{bmatrix}x & 0\\0 & y\end{bmatrix}$
σ	$\begin{bmatrix}0 & 1\\0 & 0\end{bmatrix}$	$\begin{bmatrix}0 & y\\x & 0\end{bmatrix}$	$\begin{bmatrix}0 & x\\y & 0\end{bmatrix}$	$\begin{bmatrix}0 & 0\\1 & 0\end{bmatrix}$

8.3 Quantification Tables

Given C_A a probability matrix for a proposition A, and C_B a probability matrix for B, such that
$C_A = \begin{bmatrix}a_y & a_1\\a_0 & a_x\end{bmatrix}$ $C_B = \begin{bmatrix}b_y & b_1\\b_0 & b_x\end{bmatrix}$
Then we can define the following:

Definition 8.3.1. $C_A A C_B B = C_A C_B AB = \begin{bmatrix} a_y b_y & a_y b_1 & a_1 b_y & a_1 b_1 \\ a_y b_0 & a_y b_x & a_1 b_0 & a_1 b_x \\ a_0 b_y & a_0 b_1 & a_x b_y & a_x b_1 \\ a_0 b_0 & a_0 b_x & a_x b_0 & a_x b_x \end{bmatrix} \text{AB}$

Definition 8.3.2. $C_A A + C_B B = \begin{bmatrix} y(a_y + b_y) & y a_y + b_1 & a_1 + y b_y & a_1 + b_1 \\ y a_y & y a_y + x b_x & a_1 & a_1 + x b_x \\ y b_y & b_1 & x a_x + y b_y & x a_x + b_1 \\ 0 & x b_x & x a_x & x(a_x b_x) \end{bmatrix}$

Definition 8.3.3. Summing the above 2 definitions,

$$C_A A \vee C_B B = C_A A + C_B B + C_A A C_B B = \begin{bmatrix} y(a_y + b_y + a_y b_y) & y a_y(1 + b_1) + b_1 & a_1 + y b_y & a_1 + b_1 + a_1 b_1 \\ y a_y & y a_y + x b_x & a_1 & a_1 + x b_x + a_1 b_x \\ y b_y & b_1 & x a_x + y b_y & x a_x + b_1 + x b_1 a_x \\ 0 & x b_x & x a_x & x(a_x + b_x + a_x b_x) \end{bmatrix}$$